## QUESTION 1

Let $S_{0, n}$ denote the NACS spot interest rate between now and $n$ six-month periods from now. It is given that $S_{0,1}=6 \%, S_{0,2}=7 \%, S_{0,3}=8 \%, S_{0,4}=9 \%$.
1.1) Consider a bond maturing in 2 years with cash flows of R100 every six months and a principal payment of R1000 at maturity. Use the given spot interest rates to calculate the price of a bond.
1.2) A contract is signed now in order to purchase a one year pure discount bond 12 months from now. Compute the forward interest rate (NACS) for this bond.

## QUESTION 2

Portfolio $A$ consists of a 1-year zero-coupon bond with a face value of R2000 and a 10-year zero-coupon bond with a face value of R6000. Portfolio $B$ consists of a 5.95 -year zero-coupon bond with a face value of R5000. The current yield on all bonds is $10 \%$ (NACC).
2.1) Show that both portfolios have the same duration.
2.2) Show that the percentage changes in the values of the two portfolios for a $0.1 \%$ per annum increase in yields are the same.

## QUESTION 3

3.1) Write the value of a caplet as an expectation under the spot measure.
3.2) Why might this be difficult to use directly for a caplet?
3.3) Propose a suitable numeraire asset to remedy this problem.
3.4) Define the measure associated with this new numeraire via its Radon Nikodym derivative.
3.5) Change the measure in the expectation in 3.1 to this new measure.

## QUESTION 4

Consider a Bond with annual coupon payments of R50, a principal payment of R1000 in 5 years, and a price of R1000. If the yield curve is flat at $10 \%$ NACC, what is the duration of the bond?

## QUESTION 5

Let $X$ an $Y$ be the prices of two traded securities, where

$$
\begin{aligned}
d V_{t} & =\mu V_{t} d t+\sigma V_{t} d B_{t} \\
d U_{t} & =\alpha U_{t} d t+\beta U_{t} d B_{t}
\end{aligned}
$$

and $\mu, \sigma, \alpha, \beta \in \mathbb{R} . B_{t}$ is a standard Brownian motion.
5.1) Show that

$$
\begin{equation*}
d\left(\frac{V_{t}}{U_{t}}\right)=\left(\mu-\alpha+\beta^{2}-\sigma \beta\right)\left(\frac{V_{t}}{U_{t}}\right) d t+(\sigma-\beta)\left(\frac{V_{t}}{U_{t}}\right) d B_{t} \tag{5}
\end{equation*}
$$

5.2) What is the correct choice for the market price of risk $\lambda$ so that $\left\{\frac{V_{t}}{U_{t}}\right\}_{t \in[0, T]}$ is a martingale?
5.3) The payoff of an option to exchange an investment asset worth $U$ for an investment asset worth $V$ is given as: $\max \left(V_{T}-U_{T}, 0\right)$. Using 6.1 and 6.2 , prove that the price of this option at $t=0$ is given by:

$$
\begin{equation*}
f_{0}=V_{0} N\left(d_{1}\right)-U_{0} N\left(d_{2}\right) \tag{7}
\end{equation*}
$$

where

$$
d_{1}=\frac{\ln \left(V_{0} / U_{0}\right)+\left((\sigma-\beta)^{2} / 2\right) T}{(\sigma-\beta) \sqrt{T}}
$$

and

$$
d_{2}=d_{1}-(\sigma-\beta) \sqrt{T}
$$

## QUESTION 6

Given a standard $m$-dimensional Brownian motion. $\underline{B_{t}}=\left(B_{t}^{(1)}, B_{t}^{(2)}, \ldots, B_{t}^{(m)}\right)$, define the processes

$$
\begin{aligned}
d X_{t} & =\mu_{t} d t+\sum_{i=1}^{m} \sigma_{t}^{(i)} d B_{t}^{(i)} \\
d Y_{t} & =\alpha_{t} d t+\sum_{i=1}^{m} \beta_{t}^{(i)} d B_{t}^{(i)}
\end{aligned}
$$

Apply the multi-dimensional Itô formula to prove the product rule

$$
d\left(X_{t} Y_{t}\right)=X_{t} d Y_{t}+Y_{t} d X_{t}+\sum_{i=1}^{m} \sum_{j=1}^{m} \sigma^{(i)} \beta^{(j)} p_{i, j} d t
$$

## QUESTION 7

Consider a 10 -month European put option on a 9.75 year bond with face value R1000. Suppose the current cash bond price is R960, the strike price is R900, the 10 -month risk-free interest rate is $10 \%$ NACC, and the volatility measure for the bond price in 10 months time is $9 \%$ per annum. The bond pays a semi-annual coupon of $10 \%$ and coupon payments of R50 are expected in 3 months and 9 months. Suppose the 3 -month risk-free interest rate is $9 \%$ NACC and the 9 -month risk-free interest rate is $9.5 \%$ NACC. Calculate the price of the option. You may assume the strike price of the option is the cash amount exchanged for the bond when the option is exercised.

## QUESTION 8

The continues version of the Ho-Lee model for the short rate process in a risk neutral world is

$$
d r_{t}=\theta(t) d t+\sigma d W_{t}
$$

where $\sigma$ is a constant and

$$
\theta(t)=F_{t}(0, t)+\sigma^{2} t,
$$

where $F(0, t)$ is the instantaneous forward rate for a maturity $t$ as seen at time zero.
8.1) Show that the bond price when the short rate follows the Ho-Lee is given by:

$$
\begin{equation*}
P(t, T)=A(t, T) e^{-r_{t}(T-t)} \tag{12}
\end{equation*}
$$

where

$$
\ln A(t, T)=\ln \frac{P(0, T)}{P(0, t)}+(T-t) F(0, t)-\frac{1}{2} \sigma^{2} t(T-t)^{2}
$$

8.2) Under the Ho-Lee model, calculate the price of a 1-year European call option on a zero-coupon bond that will mature in 5 years when the term structure is flat at $10 \%$, the principal of the bond is $R 100$, the strike price is $R 68$ and $\sigma=0.01$.

