Question 1 (Marks: 18)

(a) The radiative transfer equation can be written in the following form:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu. \tag{1}$$

(i) Rewrite the transfer equation with intensity as a function of optical depth, τ_ν i.e.

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu.$$

[2]

(ii) What is the name of the function provided by S_ν? What is its mathematical definition? [2]

(iii) Derive the formal solution for the radiative transfer equation shown in equation 2 below. [8]

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu \tag{2}$$

(b) (i) What is the probability that a photon will travel to an optical depth of τ_ν? [1]

(ii) Using the probability found in 1.(b)(i), write down an integral which defines the mean optical depth, $\langle\tau_\nu\rangle$ travelled by a photon. [3]

(iii) Briefly explain the concept of *mean free path* using the equation

$$\langle\tau_\nu\rangle = \alpha_\nu l_\nu = 1$$

as well as an illustrative example involving a photon moving through a cloud of absorbing gas. [3]

Question 2 (Marks: 14)

(a) Briefly explain, in words, the difference between blackbody radiation and thermal radiation in words using the concepts of intensity (I_ν), source function (S_ν) and Planck function (B_ν). [4]

(b) Derive the Rayleigh-Jeans law from $B_\nu(T)$. Name the portion of the electromagnetic spectrum where this law is most applicable. [4]

(c) A supernova remnant detected with the MeerKAT L-band receiver (central frequency, $\nu_0 = 1.28$ GHz) and has a brightness temperature of T = 4600 K. What is the specific intensity of this source? [3]

(d) B_{12} and B_{21} are the Einstein coefficients for absorption and stimulated emission. The transition probabilities of (i) absorption and (ii) stimulated emission depend on $\bar{J}$.

(i) Write down mathematical statements for the transition probabilities of (i) and (ii). [1]

(ii) Briefly explain why the transition probabilities for (i) and (ii) depend on $\bar{J}$. [2]

Question 3 (Marks: 10)

The following are Maxwell's equations in a vacuum containing no charges or current such that $\rho = 0$ and $\vec{\mathbf{J}} = 0$:

$$\nabla \cdot \vec{\mathbf{E}} = 0 \qquad \nabla \times \vec{\mathbf{E}} = -\frac{1}{c}\frac{\partial \vec{\mathbf{B}}}{\partial t}$$
$$\nabla \cdot \vec{\mathbf{B}} = 0 \qquad \nabla \times \vec{\mathbf{B}} = \frac{1}{c}\frac{\partial \vec{\mathbf{E}}}{\partial t}$$

The above four fundamental principles (Maxwell's equations) that lead to the following solutions which describe travelling waves,

$$\vec{\mathbf{E}} = \hat{a}_1 E_0 e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t}$$

and

$$\vec{\mathbf{B}} = \hat{a}_2 B_0 e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t}.$$

(a) Name all four of Maxwell's equations alongside their corresponding equations (for a medium within which the current and charge densities are non-zero). [2]

(b) Derive expressions for E_0 and B_0. [6]

(c) Show that $\omega = ck$ and $E_0 = B_0$. [2]

Question 4 (Marks: 13)

The Lorentz transformations, and their inverse, relate space and time co-ordinates between two reference frames K and K' moving with a constant speed v relative to each other along the x-axis using the following equations,

$$t' = \gamma(t - vx'/c^2),\ x' = \gamma(x - vt),\ y' = y,\ z' = z$$

$$t = \gamma(t' + vx'/c^2),\ x = \gamma(x' + vt'),\ y = y',\ z = z'$$

Here, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the Lorentz factor which is defined as $\beta = v/c$

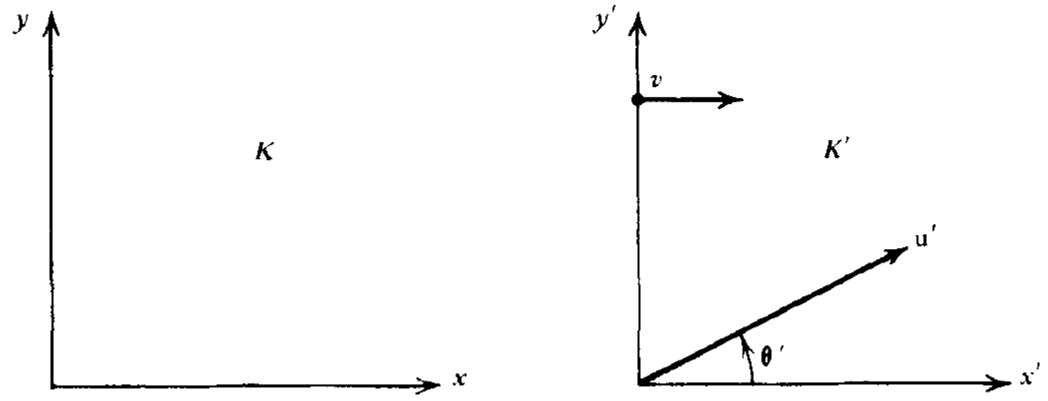

Figure 1: A particle has a velocity of u' within the K' frame which advances at velocity of $+v$ in the x-direction relative to the K frame.

(a) If the velocity of a particle is $\mathbf{u}'$ in the K' frame, as shown in Figure 1, derive expressions for the velocity components,

$$u_{\parallel} = \frac{u'_{\parallel}}{1 + u'_{\parallel}v/c^2}$$

and

$$u_{\perp} = \frac{u'_{\perp}}{\gamma(1 + u'_{\parallel}v/c^2)}.$$

which are parallel and perpendicular to $\vec{v}$ in the K frame, respectively. Furthermore, $u'_{\parallel}$ and $u'_{\perp}$ are the particle's velocity components parallel and perpendicular to $\vec{v}$ in the K' frame. [5]

(b) Show that the transformation of acceleration is,

$$a_x = \frac{a'_x}{\gamma^3\sigma^3},$$

$$a_y = \frac{a'_y}{\gamma^2\sigma^2} - \frac{u'_y v}{c^2}\frac{a'_x}{\gamma^2\sigma^3},$$

$$a_z = \frac{a'_z}{\gamma^2 \sigma^2} - \frac{u'_z v}{c^2} \frac{a'_x}{\gamma^2 \sigma^3},$$

where

$$\sigma = 1 + \frac{v u'_x}{c^2}$$

[8]

Question 5 (Marks: 10)

A particle moving relativistically emits radiation with an angular distribution of power within the instantaneous rest-frame K' according to the equation:

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{(\gamma^2 a_\parallel^2 + a_\perp^2)}{(1-\beta\mu)^4} \sin^2 \Theta' \tag{3}$$

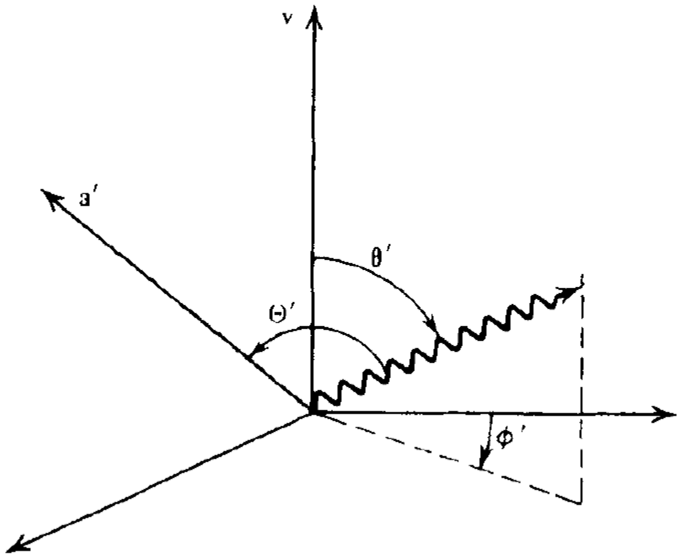

Figure 2: The dipole radiation pattern for a particle at instantaneous rest within frame K'.

(a) Derive the equation for the angular distribution of power when the acceleration is parallel to the velocity such that

$$\sin^2 \Theta' = \frac{\sin^2 \theta}{\gamma^2 (1-\beta\mu)^2}.$$

[3]

(b) Derive the equation for the angular distribution of power when the acceleration is perpendicular to the velocity such that

$$\sin^2\Theta' = 1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\mu)^2}.$$

[3]

(c) Draw a diagram showing the dipole pattern of radiation emitted by a particle when:

(i) the particle is at rest. [2]

(ii) the particle with a parallel acceleration ($a_{\|}$) and velocity. [2]

Question 6 (Marks: 15)

(a) Draw a detailed diagram of a relativistic electron accelerating in a helical path within a uniform magnetic field. Label the velocity vector (and its components) as well as the acceleration. [4]

(b) Derive equations for the parallel and perpendicular acceleration of the electron as it accelerates in the helical trajectory shown in your diagram [3]

(c) Write the Compton scattering equation

$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1-\cos\theta)}$$

in terms of wavelength. [6]

(d) Write down the equation for the Compton wavelength. [1]

(e) Re-write the equation you obtained in 6.(c) to include the Compton wavelength. [1]

BONUS QUESTION:

(f) Assuming that an electron, with Lorentz factor of $\gamma = 10^8$ is immersed in a radiation field of CMB (cosmic microwave background) photons that emit radiation as a Planck distribution of have temperature $T = 2.73$ K. A magnetic field of strength $B = 10^5$ Gauss passes through the radiation field which has an initial photon energy density of $U_{\rm ph} = 10^{28}$ erg cm^{-3}.

(i) Calculate the Compton power, P_{Compt}, of the radiation scattered by the electron. [4]

(ii) Draw a diagram showing the scattering geometry for incident and resultant Compton emission in the rest frame of the observer (frame K) and (ii) the rest frame of the electron (frame K'). [4]

(iii) What is the ratio of the Compton power, P_{Compt}, to the synchrotron power, P_{synch}? [2]

EQUATION SHEET:

Equations

$F(r) = \text{const.}/\text{r}^2$
$F = \int I \cos\theta d\Omega$
$F = \pi B$
$u(T) = aT^4$
$F = \sigma T^4$
$L = 4\pi R^2 F$
$S_\nu = j_\nu/\alpha_\nu$
$dQ = dU + pdV$
$dS = \frac{dQ}{T}$
$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp{(h\nu/kT)}-1}$
$B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp{(hc/\lambda kT)}-1}$
$\int_0^\infty \phi(\nu)d\nu = 1$
$\bar{J} \equiv \int_0^\infty J_\nu \phi(\nu)d\nu$
$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21}\phi(\nu)$
$\alpha_\nu = \frac{h\nu}{4\pi}\phi(\nu)(n_1 B_{12} - n_2 B_{21})$
$\frac{n_1}{n_2} = \frac{g_1}{g_2}\exp{(h\nu_0/kT)}$

$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \frac{\vec{\mathbf{v}}}{c} \times \vec{\mathbf{B}}\right)$
$\nabla \cdot \vec{\mathbf{D}} = 4\pi\rho$
$\nabla \cdot \vec{\mathbf{B}} = 0$
$\nabla \times \vec{\mathbf{E}} = -\frac{1}{c}\frac{\partial \vec{\mathbf{B}}}{\partial t}$
$\nabla \times \vec{\mathbf{H}} = \frac{4\pi}{c}\vec{\mathbf{j}} + \frac{1}{c}\frac{\partial \vec{\mathbf{D}}}{\partial t}$
$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$
$\vec{\mathbf{B}} = \mu \vec{\mathbf{H}}$
$\epsilon = \mu = 1$ (in vacuo)

$\beta = v/c$
$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
$m_e = 9.1 \times 10^{-28}$ g
$q_e = 4.8 \times 10^{-10}$ esu
$L = L_0/\gamma$
$T = \gamma T_0$

$P_{\text{Compt}} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_{\text{ph}}$
$P_{\text{synch}} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_B$

$(fg)\prime = f\prime g + fg\prime$
$(\frac{f}{g})\prime = \frac{f\prime g - fg\prime}{g^2}$

Constants

$c = 2.998 \times 10^{10}$ cm s^{-1}
$h = 6.626 \times 10^{-27}$ erg s
$k = 1.381 \times 10^{-16}$ erg K^{-1}
$\sigma = 5.67 \times 10^{-5}$ erg cm^{-2} K^{-4} s^{-1}
$a = \frac{4\sigma}{c} = 7.56 \times 10^{-15}$ erg cm^{-3} K^{-4}
$\mathrm{R}_\odot = 695\ 500$ km
$\mathrm{L}_\odot = 3.85 \times 10^{33}$ erg s^{-1}

TOTAL: 80 marks

DURATION: 2 hours