Question 1 [12 marks]

For questions 1.1 – 1.12, choose the correct answer, and make a cross (X) in the corresponding block.

Question	**a**	**b**	**c**	**d**	**e**
1.1					
1.2					
1.3					
1.4					
1.5					
1.6					
1.7					
1.8					
1.9					
1.10					
1.11					
1.12					

1.1 Select the formula that is the negation of $p \rightarrow (q \vee r)$. (1)

(a) $\neg p \rightarrow \neg(q \vee r)$

(b) $p \wedge \neg(q \vee r)$

(c) $\neg p \wedge (q \vee r)$

(d) $\neg(q \vee r) \rightarrow \neg p$

(e) None of the above.

1.2 Select the correct natural language translation of the predicate formula: (1)

$$(\forall y \in \mathbb{Q})(y^2 < 0)$$

(a) The square of any rational number is positive.

(b) There are rational numbers with negative squares.

(c) Every rational number has a negative square.

(d) Some rational numbers are less than zero.

(e) None of the above.

1.3 If $f(x) = \dfrac{1}{x-3}$ and $g(x) = 2x + 4$, then $g \circ f =$: (1)

(a) $\dfrac{1}{2x+1}$

(b) $x - 3$

(c) $\dfrac{2x+4}{x-3}$

(d) $\dfrac{2}{x-3} + 4$

(e) None of above.

1.4 Consider the summation: $\frac{7}{8} - \frac{8}{9} + \frac{9}{10} - \frac{10}{11} + \frac{11}{12} - \frac{12}{13}.$

Select the correct sigma notation representation of the above summation: (1)

(a) $\displaystyle\sum_{i=0}^{6} \frac{7+i}{8+i}(-1)^i.$

(b) $\displaystyle\sum_{i=1}^{6} \frac{7+i}{8+i}(-1)^{i+1}.$

(c) $\displaystyle\sum_{i=0}^{5} \frac{7+i}{8+i}(-1)^{i+2}.$

(d) $\displaystyle\sum_{i=1}^{6} \frac{7+i}{8+i}(-1)^i.$

(e) None of the above

1.5 Consider the following four functions: (i) $f(x) = \dfrac{\sqrt{x}+1}{x-1}$ (ii) $f(x) = \dfrac{ex^2+1}{x+3}$

(iii) $f(x) = \dfrac{x^e + e^x}{x^2+3x+1}$ (iv) $f(x) = \dfrac{\sqrt{2}x^3 - 2x^2 + 3}{x^2+x-1}$

Which of the functions above are rational functions? (1)

(a) (ii), (iii) and (iv) only
(b) (ii) and (iv) only
(c) (i) and (iii) only
(d) (ii) only
(e) None of the above combinations are correct.

1.6 The domain of the function $y = \arctan x$ is: (1)

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(b) $(-\infty, \infty)$
(c) $[0, \pi]$
(d) $[-1, 1]$
(e) None of the above.

1.7 $\dfrac{d}{dx}\left(10^{\cot x}\right) =$ (1)

(a) $(\ln 10)\, 10^{\csc^2 x}$
(b) $(\ln 10)\, 10^{-\csc^2 x}$
(c) $(\ln 10)\, 10^{\cot x} \csc^2 x$
(d) $-(\ln 10)\, 10^{\cot x} \csc^2 x$
(e) None of the above.

1.8 Differentiate $y = \log_5 (1 + 2x)$. (1)

(a) $y' = \dfrac{1}{(1+2x)(\ln 5)}$

(b) $y' = \dfrac{2}{(1+2x)(\ln 5)}$

(c) $y' = \dfrac{2\ln 5}{1+2x}$

(d) $y' = \dfrac{1+2x}{\ln 5}$

(e) None of the above.

1.9 Calculate y' if $y = x \tan^{-1}(4x)$. (1)

(a) $y' = \dfrac{4x}{1+16x^2} + \tan^{-1}(4x)$

(b) $y' = \dfrac{x}{1+16x^2} + \tan^{-1}(4x)$

(c) $y' = \dfrac{4}{1+16x^2} + x\tan^{-1}(4x)$

(d) $y' = \dfrac{1}{1+16x^2} + (4x)\tan^{-1}(4x)$

(e) None of the above.

1.10 Suppose $F(x)$ is an antiderivative of $f(x)$ and an antiderivative of $g(x)$. Which of the following statements are always true? (1)

(i) $g'(x) = F(x)$ and $f'(x) = F(x)$.

(ii) $f'(x) = g(x)$

(iii) $g(x) = f(x)$.

(iv) $F'(x) = g(x)$ and $F'(x) = f(x)$

(a) Only (i).

(b) (i) and (ii).

(c) (i) and (iii).

(d) (ii) and (iv).

(e) (iii) and (iv).

1.11 Find the value of the integral: $\displaystyle\int_1^4 (1+2x)\,dx$. (1)

(a) 18

(b) 12

(c) 9

(d) 6

(e) None of the above.

1.12 Select an appropriate substitution that can be used to evaluate the indefinite integral $\int \tan x \ln(\cos x)\, dx$ and will yield a polynomial integrand. (1)

(a) $u = \tan x$

(b) $u = \ln(\cos x)$

(c) $u = \cos x$

(d) $u = \sin x$

(e) None of the above.

Question 2 [2 marks]

The solution set for the inequality $|x - b| \geqslant d$ is $(-\infty, -2] \cup [20, \infty)$. Calculate the values of b and d. (2)

Question 3 [2 marks]

If $\sin\theta + \cos\theta = \sqrt{2}$ prove that $\sin 2\theta = 1$. [Hint: Square both sides of the given equation.] (2)

Question 4 [2 marks]

Give a counterexample to show that the following equation is not always true. You must show that the equation fails for your counterexample. (2)

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2.$$

Question 5 [3 marks]

Complete the truth table for the formula $p \to (q \vee \neg p)$ and then determine whether it is a tautology. (3)

p	q	

Question 6 [2 marks]

Write the following statement in predicate language: (2)

Every real number is negative and greater than π.

Question 7 [2 marks]

Prove that if $x \in \mathbb{Z}$ and $3 \mid x$, then $9 \mid (x^2 - 12x)$. (2)

Question 8 [3 marks]

Use known logical equivalences to show that $\neg(p \to \neg(q \vee \neg p)) \equiv p \wedge q$. (3)

Question 9 [2 marks]

Let $h(x) = \dfrac{1}{\sqrt{x^2 - 6x - 7}}$. Find the domain of h and give your answer in interval notation. (2)

Question 10 [2 marks]

Sketch the graph of the function $f(x) = (x-1)^2 + 1$ using the appropriate transformations. Your answer should include 3 sketches. Indicate all intercepts clearly. (2)

Question 11 [2 marks]

Simplify: $\sec\left(\arctan\left(-\frac{4}{3}\right)\right)$. (2)

Question 12 [2 marks]

Find the inverse of the function $f(x) = x^2,\ x \leqslant 0$, and write it in the form $f^{-1}(x)$. (2)

Question 13 [3 marks]

Show that the equation $0 = x^3 - x - 1$ has a root in the interval $(1, 2)$. (3)

Question 14 [4 marks]

Consider the function:

$$f(x) = \begin{cases} x^2 - 8 & \text{if } x < 3 \\ x^3 - 27 & \text{if } x \geqslant 3 \end{cases}$$

(a) Find

(i) $\lim_{x \to 3^-} f(x)$ (1)

(ii) $\lim_{x \to 3^+} f(x)$ (1)

(b) What does this tell us about $\lim_{x \to 3} f(x)$? (1)

(c) Is $f(x)$ continuous at $x = 3$? Motivate your answer. (1)

<u>Question 15</u> [5 marks]

(a) Prove the following theorem: (4)

If a function f is differentiable at $x = a$, then it is continuous at a.

(b) Is the converse of the above theorem true? Justify your answer. (1)

Question 16 [2 marks]

Find the horizontal asymptotes of the curve $y = \dfrac{2e^x}{e^x - 5}$. (2)

Question 17 [2 marks]

Find $\displaystyle\lim_{x \to 0^+} [\ln x - \ln(\sin x)]$ without using L'Hospital's Rule. (2)

Question 18 [4 marks]

Find $\dfrac{dy}{dx}$ if $e^{\frac{x}{y}} = x^4 + y^4.$ (4)

Question 19 [3 marks]

Use logarithmic differentiation to find y' if: $y = x^{\cosh x - 1}.$ (3)

Question 20 [3 marks]

Evaluate: $\lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$. (3)

Question 21 [2 marks]

Let $F(x) = \int_2^x \frac{1}{t^2}\, dt$. Find an equation of the tangent line to the curve $y = F(x)$ at the point with x-coordinate 2. (2)

Question 22 [2 marks]

Evaluate the indefinite integral: $$\int \left(\frac{1}{\sqrt{1-x^2}} - \frac{1}{x} + \cosh x \right) dx.$$ (2)

Question 23 [4 marks]

Evaluate the definite integral: $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 x \sin x \, dx.$$ (4)