Question 1 [8]

The box below contains possible answers to the following questions. Fill in the blank spaces with the correct answer.

$\ln 2 + 1$; $-\sqrt{\frac{3}{2}}$; $-\ln 3$; $\mathbf{0}$; $\ln 2 - 1$; $\sqrt{\frac{3}{2}}$; $-4\mathbf{i}$; -3 ; $1 - \ln 3$; -4 ; 18π ; 3 ; 0; NOTA

(1.1) If $\mathbf{F}(x, y, z) = z^2 y \sin x\mathbf{i} - z^2 \cos x\mathbf{j} - 2zy \cos x\mathbf{k}$, then curl $\mathbf{F}$ at $(0, 1, 2)$ is ——. (3)

(1.2) If $f(x, y, z) = x \sin(yz)$, then the directional derivative of f at $(1, 3, 0)$ in the direction of $\mathbf{v} = \mathbf{i}+2\mathbf{j}-\mathbf{k}$ is —— . (2)

(1.3) Let $f(x, y) = e^{xy} + \ln\left(\frac{1}{x}\right)$ and let C be a semicircular path in the upper half of the plane from $(1, 0)$ to $(3, 0)$. The value of the following integral, $\int_C \nabla f \cdot d\mathbf{r}$ equals ——- (3)

Question 2 [10]

Choose the correct option for the multiple choice questions below and write your answer in the table provided.

Question	Answer
1.1	
1.2	
1.3	
1.4	
1.5	

(1.1) The limit of the following function

$$f(x, y) = \frac{xy + yz}{x^2 + y^2 + z^2}$$

as (x, y, z) approaches $(0, 0, 0)$ (2)

(a) equals 0 (b) equals $\frac{1}{2}$ (c) equals $-\frac{1}{2}$ (d) does not exist

(1.2) The line integral $\oint_C y^3 \mathrm{d}x - x^3 \mathrm{d}y$ where C is the circle $x^2 + y^2 = 4$ (2)

(a) equals -12π (b) equals -24π (c) equals 24π (d) equals 18π (e) equals -18π

(1.3) The quadratic surface represented by the equation $x^2 + 6z + 4x + y^2 - z^2 = 5$ is . (2)

(a) an ellipsoid (b) a cone (c) an elliptic paraboloid (d) a hyperbolic paraboliod

(1.4) The domain and range of the following function

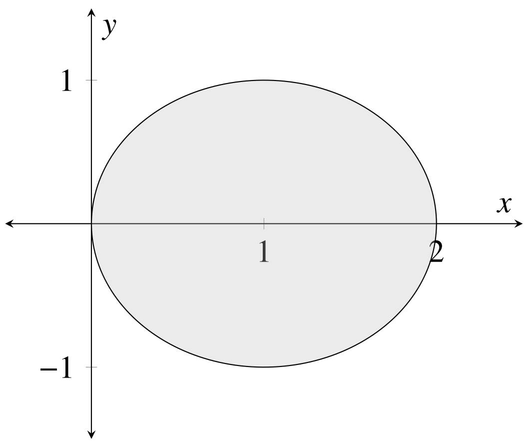
$$f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$$

is given by (2)

(a) $\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}; z \in [0, 6]$ (b)$\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\}; z \in [0, 6]$

(c) $\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} \geq 1\}; z \in (0, 6)$ (d)$\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{9} + \frac{y^2}{4} \geq 1\}; z \in (0, 6)$

(1.5) An iterated integral which represents the area of the region below is given by: (2)



(a) $2\int_0^{2\pi}\int_0^{2\sin\theta} r\,drd\theta$ (b) $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} drd\theta$ (c) $2\int_0^{\pi/2}\int_0^{2\cos\theta} r\,drd\theta$ (d) $\int_{-\pi/2}^{\pi/2}\int_0^{2\sin\theta} r\,drd\theta$

Question 3 [6]

Determine whether the following statements are true or false. If true, give a short justification. If false, explain why or give a counter example.

(3.1) A critical point is either a local minimum or a local maximum. (1)

(3.2) The function

$$f(x,y) = \begin{cases} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$. (2)

(3.3) For a function f to be continuous at (a,b) it is sufficient to check that f is defined at (a,b) and that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists. (1)

(3.4) If $\int_0^{100} e^{-x^2}\,dx = I$, then $\int_0^{100}\int_0^{100} e^{-x^2}e^{-y^2}\,dy\,dx = I^2$. (1)

(3.5) For any vector fields $\mathbf{F}(x,y,z)$ and $\mathbf{G}(x,y,z)$, we have $\text{curl}(\mathbf{F}+\mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$. (1)

Question 4 [6]

Instruction: Answer only ONE question from the following. Show full working of your solution.

- Use the Lagrange Multiplier method to find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.
- Suppose $z = f(x, y)$, where $x = u + v$ and $y = u^2v$. Find

$$\frac{\partial^2 z}{\partial u \partial v}$$

.

Extra Page

Question 5 [5]

Instruction: Answer only ONE question from the following. Show full working of your solution.

- Evaluate the integral $\iint_R \sqrt{x^2 + y^2}\, dA$, where R is the part of the unit disk in the first quadrant.
- Use a double integral and a convenient coordinate system to find the volume of the solid bounded by $z = e^{-x^2}$ and the planes $y = 0$, $y = x$, and $x = 1$.

Question 6 [5]

Convert the following integral to cylindrical **or** spherical coordinates:

$$\int\int\int_E \sqrt{x^2+y^2}\,\mathrm{d}V$$

where E is the solid bounded by the graph $4x^2+4y^2+z^2=16$ and below by the graph $z=\sqrt{4x^2+4y^2}$. Answer only one. Show all calculations and sketches where necessary.

Question 7 [7]

Use a suitable change of variables to evaluate the integral

$$\iint_D \left(\sqrt[3]{x^2y}.\sqrt[3]{y^2x^{-2}}\right)\, dA,$$

where D is the region bounded by the four curves $x^2y = 4$, $x^2y = 9$, $y = x$ and $\frac{y}{x} = 2$.

Question 8 [5]

Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equations $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$.

Question 9 [8]

(9.1) Show that if

$$\mathbf{F}(x, y, z) = \sqrt{y^4} z^3 \mathbf{i} + \sqrt{4x^2y^2} z^3 \mathbf{j} + \sqrt{9x^2y^4z^4} \mathbf{k}$$

then curl $\mathbf{F} = \mathbf{0}$. (3)

(9.2) If $\mathbf{F}$ is conservative, find a scalar function f such that $\mathbf{F} = \nabla f$. (2)

(9.3) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. If $\mathbf{F} = \frac{\mathbf{r}}{r^p}$. Find div $\mathbf{F}$. (3)