Question 1 [10]

Determine whether the following statements are TRUE or FALSE. Motivate the statement if TRUE; provide a counterexample if FALSE.

(a) If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space. (2)

(b) If A is a square matrix and $A\overline{x} = \lambda\overline{x}$ for some nonzero scalar λ, then $\overline{x}$ is an eigenvector of A. (2)

(c) If A is an $n \times n$ matrix, then $\langle \overline{u}, \overline{v} \rangle = A\overline{u} \cdot A\overline{v}$ defines an inner product on $\mathbb{R}^n$. (2)

(d) If A is a square matrix, then AA^T and A^TA are orthogonally diagonalizable. (2)

(e) There is exactly one linear transformation $T : V \to W$ such that $T(-\overline{v}) = -T(\overline{v})$ for all $\overline{v}$ in V. (2)

Question 2 [3]

Find the rank of A and nullity of A^T where

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \\ 2 & 2 & 6 & 8 \end{bmatrix}.$$

Question 3 [4]

Let A be an $n \times n$ matrix. Prove that λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0.$$

Question 4 [2]

Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3.$$

(a) What can you say about the dimensions of the eigenspaces of A?

(b) If A is diagonalizable, what can you say about the dimensions of the eigenspaces of A?

Question 5 [4]

Let $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$ be in $\mathbb{R}^2$. Show whether or not

$$\langle \overline{x}, \overline{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$$

is an inner product on $\mathbb{R}^2$.

Question 6 [3]

Let W be a subspace of an inner product space V. Show that $W^{\perp}$ is a subspace of V.

Question 7 [2]

Prove, or disprove, that if A is an $m \times n$ matrix such that $A^T A = I$ then A is orthogonal.

Question 8 [3]

Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables.

$$Q = 5x_1^2 + 4x_1x_2 + 2x_2^2$$

Question 9 [4]

Determine whether the following transformations are linear. If they are, prove it, if not provide a counterexample to one of the properties.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, with $T(x, y) = (x^2, y^2)$.

(b) Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, where addition and scalar multiplication are defined pointwise, i.e. $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$. Then, let $T : V \to \mathbb{R}^2$, with

$$T(f) = \begin{bmatrix} f(0) \\ f(1) + 1 \end{bmatrix}.$$

Question 10 [5]

Prove that every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Question 11 [2]

Let $T : \mathcal{P}_1 \to \mathbb{R}^2$ be the transformation defined by the formula

$$T(p(x)) = (p(0), p(1)).$$

Find $T(1-2x)$ and $T^{-1}((2,3))$.

Question 12 [3]

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ be the matrix for $T : \mathcal{P}_1 \to \mathbb{R}^2$ with respect to the bases $B = \{1 + x, 1 - x\}$ and $C = \{(1, 1), (-1, 1)\}$.

(a) Find $[T(-1 - 3x)]_C$.

(b) Hence, find $T(-1 - 3x)$.