Question 1 [10]

Determine whether the following statements are TRUE or FALSE. Motivate the statement if TRUE; provide a counterexample if FALSE.

(a) If $\text{rank}(A^T) = \text{rank}(A)$, then A is square. (2)

(b) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible. (2)

(c) If $\langle \overline{u}, \overline{v} \rangle = 0$, then $\overline{u} = \overline{0}$ or $\overline{v} = \overline{0}$. (2)

(d) Every orthogonal matrix is orthogonally diagonalizable. (2)

(e) There is a subspace of $M_{2,3}$ (the space of 2×3 matrices) that is isomorphic to $\mathbb{R}^4$. (2)

Question 2 [3]

Let W be the subspace of $\mathbb{R}^3$ spanned by the vectors $\overline{v}_1 = (1, 2, 0)$ and $\overline{v}_2 = (0, 1, 1)$. Find a basis for $W^\perp$.

Question 3 [3]

Prove that if k is a positive integer, λ is an eigenvalue of a matrix A, and $\overline{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\overline{x}$ is a corresponding eigenvector.

Question 4 [3]

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

then compute the resulting diagonal matrix D.

Question 5 [4]

Let $\mathcal{P}_3$ have the evaluation inner product at the sample points

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1 \quad \text{and} \quad x_3 = 2.$$

Find $\langle \overline{p}, \overline{q} \rangle$ and $\|\overline{p}\|$, for $\overline{p} = x + x^3$ and $\overline{q} = 1 + x^2$.

Question 6 [4]

State and prove the Generalized Theorem of Pythagoras.

Question 7 [2]

Find values for a, b and c such that the matrix

$$\begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is orthogonal.

Question 8 [3]

Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables.

$$Q = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$$

Question 9 [3]

Fix an $m \times n$ matrix A. Determine whether the operator $T : M_{l,m} \to M_{l,n}$ such that

$$T(B) = BA$$

is linear. If it is, prove it, if not then provide a counterexample to one of the properties.

Question 10 [4]

Let V and W be finite-dimensional real vector spaces with the same dimension, and let $T : V \to W$ be a linear transformation. Prove that $\ker(T) = \{\overline{0}\}$ if and only if T is onto.

Question 11 [2]

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear operator defined by the formula

$$T(x_1, x_2, \ldots, x_n) = (a_1x_1, a_2x_2, \ldots, a_nx_n)$$

where $a_1, \ldots, a_n$ are constants.

(a) Under what conditions on $a_1, \ldots, a_n$ will T need in order to have an inverse?

(b) Assuming the conditions is (a) are met, find a formula for T^{-1}.

Question 12 [4]

Let $T_1 : \mathcal{P}_1 \to \mathcal{P}_1$ be defined by
$$T_1(p(x)) = xp'(x),$$
and let $T_2 : \mathcal{P}_1 \to \mathcal{P}_2$ be defined by
$$T_2(q(x)) = xq(x).$$
Fix the bases $B = \{1, 1+x\}$ and $B' = \{1, 1+x, 1+x+x^2\}$ for $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

(a) Find $[T_1]_B$ and $[T_2]_{B',B}$.

(b) Use (a) to determine whether or not T_1 is invertible. If it is, find $[T_1^{-1}]_B$.