APM8X20 Lie Groups and Lie Algebras Examination

Time: 3hrs 60 Marks is Full Marks

Question 1 [15]

(i) Explain, without using mathematical terminology, what is a Lie Group and a Lie algebra.
(ii) Give an example of a simple Lie group.
(iii) What is the relationship between a Lie group and a Lie algebra?

Question 2 [10]

From the article, “Applications of Lie Groups and Lie Algebra to Computer Vision: A Brief Survey”, explain why Lie groups and Lie algebras are useful for pattern recognition.

Question 3 [5]

Calculate the generator of the group of rotations about the y-axis.

Question 4 [10]

Determine the Lie algebra associated with the generators

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \;\; \mathrm{X}_2 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Show all working.

Question 5 [15]

The coefficients of the extended or prolonged generator of Lie point symmetries

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y'} + \eta^{(2)}\frac{\partial}{\partial y''} + \eta^{(3)}\frac{\partial}{\partial y'''}$$

is given by

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2; \tag{3.20}$$

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + \{\eta_y - 2\xi_x - 3\xi_y y'\}y''; \tag{3.21}$$

$$\eta^{(3)} = \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 + 3\{\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2\}y'' - 3\xi_y y''^2 + \{\eta_y - 3\xi_x - 4\xi_y y'\}y'''. \tag{3.22}$$

Calculate the determining equations necessary to solve

$$y'' + y^3 = 0.$$

Show all working.

Question6 [10]

Calculate a group variant solution, y=Φ(x), corresponding to the Lie symmetry

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$