APPLIED MATHEMATICS

Introduction to Differential Equations
APM02A2/APM2A10

Duration: 180 minutes **Marks:** 100
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Moderator: Prof E Momoniat

Instructions:

1. Indicate your ASSIGNED NUMBER on the top right of all scripts submitted and ENCIRCLE it.
2. Check that this question paper consists of 2 pages in total, including this one.
3. Carefully read and follow the instructions of each question.
4. Answer the questions in order. Cross out answers that are not to be marked.
5. All calculations must be shown. Solutions written in PENCIL will NOT be considered.
6. Pocket calculators are permitted.

Question 1 (20 marks)

(a) Consider the differential equation (DE) $y' = f(x, y)$ where

$$f(x, y) = \sqrt{3y - x}$$

(i) Without solving the DE, plot direction fields on the x-axis, the y-axis, and $y = x$. Sketch a graph of any solution curve above $y = x$.

(ii) For what choices of (x_0, y_0) will the initial value problem

$$y' = \sqrt{3y - x}, \quad y(x_0) = y_0$$

have a unique solution in an open interval (a, b) that contains x_0?

(b) Given the differential equation (DE)

$$e^{5x/2}(5xy + 2y + 5)dx + 2xe^{5x/2}dy = 0,$$

show that the DE is exact. Use the method for exact equations to solve the DE.

Question 2 (20 marks)

Consider the initial value problem (IVP) given by

$$y'' - 4y = e^{2x}, \; y(0) = 1, \; y'(0) = -4. \quad (*)$$

(a) Give the two IVPs of the associated homogeonous DE ($y_h(x)$) and nonhomogeneous DE ($y_{nh}(x)$) needed to solve ($*$).

(b) Solve the IVP (*) and use Green's function to solve for $y_{nh}(x)$.

Question 3 (13 marks)
Solve the Cauchy-Euler equation

$$x^2y'' + xy - y = \frac{1}{x+1}, \quad x > 0$$

Question 4 (12 marks)
The motion of a certain spring-mass system is governed by the differential equation

$$\frac{d^2x}{dt^2} + 0.125\frac{dx}{dt} + x = 0,\ x(0) = 2,\ x'(0) = 0.$$

(a) Describe fully the motion represented by the differential equation. (2)

(b) Calculate the spring constant, damping constant and the mass of the object. (4)

(c) The position of the mass at any time t is given by (6)

$$x(t) = e^{-\frac{t}{16}}\left(2\cos\left(\frac{\sqrt{255}}{16}t\right) + \frac{2}{\sqrt{255}}\sin\left(\frac{\sqrt{255}}{16}t\right)\right).$$

(i) Find the damped amplitude, quasi-period and quasi-frequency of the motion.

(ii) Find the first time at which the mass first passes through its equilibrium position.

Question 5 (13 marks)
Given

$$f(t) = \begin{cases} 1, & 0 \leqslant t < 1 \\ t, & t \geqslant 1 \end{cases}$$

(a) Use the definition of the Laplace transform to find the Laplace transform of f.

(b) Express $f(t)$ in terms of the unit step function.

Question 6 (22 marks)

(a) Find the Laplace transform $Y(s)$ of the differential equation (5)

$$-ty'' - y' + 4ty = 0,\ y(0) = 4,\ y'(0) = 0.$$

(b) Use the Convolution theorem to find the Laplace transform of = (2)

$$\cos(-4t) \star \mathcal{U}(t - \frac{\pi}{3})$$

(c) Use Laplace transforms to solve the system of differential equations. (15)

$$\begin{aligned} x' + 3x &= 1 - y', \\ y' - y &= e^t - x' + x, \\ & x(0) = 0 = y(0). \end{aligned}$$

END OF QUESTION PAPER