Exam

1. $\left(\tan^2 x - \frac{y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2} - e^y\right) dy = 0$

$M(x,y) = \tan^2 x - \frac{y}{x^2+y^2}$; $N(x,y) = \frac{x}{x^2+y^2} - e^y$

$$\frac{\partial M}{\partial y} = \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{x^2+y^2-x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial M}{\partial y}$$

$\therefore$ The DE is exact.

Since the DE is exact, there exist a function f of x and y such that

$$\frac{\partial f}{\partial x} = M(x,y) \text{ and } \frac{\partial f}{\partial y} = N(x,y)$$

That is, $\frac{\partial f}{\partial x} = \tan^2 x - \frac{y}{x^2+y^2}$ and $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} - e^y$

* Then $f(x,y) = \int \frac{\partial f}{\partial x} dx = \int \left(\tan^2 x - \frac{y}{x^2+y^2}\right) dx$ *

$$= \tan x - x - \tan^{-1}\left(\frac{x}{y}\right) + \varphi(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} + \varphi'(y) = \frac{x}{x^2+y^2} - e^y$$

$$\Rightarrow \varphi'(y) = -e^y$$

$$\therefore \varphi(y) = -e^y$$

Thus, the general solution is

$$\tan x - x - \tan^{-1}\left(\frac{x}{y}\right) - e^y = C$$

OR

* $f(x,y) = \int \frac{\partial f}{\partial y} dy = \int \left(\frac{x}{x^2+y^2} - e^y\right) dy$

$$= \tan^{-1}\left(\frac{x}{y}\right) - e^y + \phi(x)$$

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2} + \varphi'(x) = \tan^2 x - \frac{y}{x^2+y^2}$$

$$\Rightarrow \varphi'(x) = \tan^2 x$$

$\Rightarrow \quad y(x) = \tan x - x$

$\therefore$ The general solution is

$\tan^{-1}(\frac{x}{y}) - e^y + \tan x - x = C$

2) $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = F(y/x)$

This is a homogeneous DE

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ✓

The equation becomes

$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x^2 v} = \frac{v^2 - 1}{2v}$ ✓

$\Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = \frac{-1-v^2}{v}$

$\Rightarrow \quad \frac{v}{1+v^2}\,dv = -\frac{dx}{x}$

$\Rightarrow \quad \ln(1+v^2) = -\ln|x| + C$ ✓

$\Rightarrow \quad 1+v^2 = \frac{A}{x}$

i.e. $\quad 1 + \frac{y^2}{x^2} = \frac{A}{x}$

$y(1) = 0 \Rightarrow 1 = A$ ✓

$\therefore \quad 1 + \frac{y^2}{x^2} = \frac{1}{x}$ ✓

$\Rightarrow \quad y^2 = x - x^2$

3) $x\frac{dy}{dx} + y = \frac{1}{y^3} \Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{1}{y^3}$

This is a Bernoulli DE with $n = -3$

Multiply by y^3: $\quad y^3\frac{dy}{dx} + \frac{y^4}{x} = 1 \qquad (\#)$

Let $v = y^4 \Rightarrow \frac{dv}{dx} = 4y^3\frac{dy}{dx}$

$(\#)$ becomes $\quad \frac{1}{4}\frac{dv}{dx} + \frac{v}{x} = 1$

$\Rightarrow \quad \frac{dv}{dx} + \frac{4v}{x} = 4$

This is a linear first order DE.

The integrating factor is $I(x) = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$

$\therefore$ The solution is

$$x^4 v = \int x^4 \cdot 4\,dx$$

$$\Rightarrow x^4 v = \frac{4}{5}x^5 + C$$

$$\Rightarrow x^4 y^4 = \frac{4}{5}x^5 + C$$

$$\Rightarrow y = \frac{4}{5}x + \frac{c}{x^4}$$

4) If P is the population and t is the time variable, then

(i) $\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$, where k is the proportionality constant.

(ii) At 10h00, $t = 0 \Rightarrow P(0) = 100\,000$

At 11h00, $t = 1 \Rightarrow P(1) = 200\,000$

Now $\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt$

$$\Rightarrow \ln P = kt + C$$

$P(0) = 100\,000 \Rightarrow \ln 100000 = C$

$\Rightarrow \quad \ln P = kt + \ln 100000$

$P(1) = 200\,000 \Rightarrow \ln 200\,000 = k + \ln 100000$

$$\Rightarrow k = \ln\left(\frac{200\,000}{100\,000}\right) = \ln 2$$

$$\therefore \ln P = (\ln 2)t + \ln 100\,000$$

$$\Rightarrow P = 100\,000\, e^{(\ln 2)t}$$

$$= 100000\, e^{0.693t}$$

(iii) $P = 500\,000$

$\Rightarrow 500\,000 = 100\,000\, e^{0.693t}$

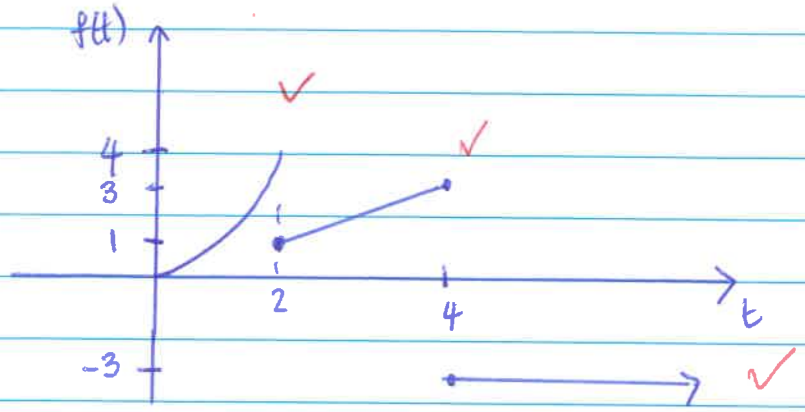
$\Rightarrow t = \frac{1}{0.693}\ln 5 = 2.322$ hours

≈ 2 hours 19 minutes

MATE2A2 Exam: MAIN EXAM MEMO

Question 5

a)



b) $f(t) = t^2\,(u(t-0) - u(t-2)) + (t-1)(u(t-2) - u(t-4)) - 3\,u(t-4)$

$= t^2\,u(t) + (t - t^2 - 1)\,u(t-2) - (t+2)\,u(t-4)$

c) $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 u(t) + (t-t^2-1)u(t-2) - (t+2)\,u(t-4)\}$

$= \mathcal{L}\{t^2 u(t)\} + \mathcal{L}\{(t-t^2-1)u(t-2)\} - \mathcal{L}\{(t+2)\,u(t-4)\}$

- $\mathcal{L}\{t^2 u(t)\} = 2/s^3$

Recall: $\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

- $\mathcal{L}\{(t-t^2-1)\,u(t-2)\}$

Here, $a = 2$ and $f(t) = t - t^2 - 1$

Hence, $f(t+2) = -t^2 - 3t - 3$

$\therefore \mathcal{L}\{(t-t^2-1)u(t-2)\} = e^{-2s}\,\mathcal{L}\{-t^2 - 3t - 3\}$

$= e^{-2s}\left(-\frac{2}{s^3} - \frac{3}{s^2} - \frac{3}{s}\right)$

- $\mathcal{L}\{(t+2)\,u(t-4)\}$

Here, $a = +4$ and $f(t) = t+2$

Hence, $f(t+4) = t + 6$

ME

$$\therefore \mathcal{L}\{(t+2)u(t-4)\} = e^{-4s}\,\mathcal{L}\{t+6\}$$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

Finally,

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right) - e^{-4s}\left(\frac{1}{s^2} + \frac{6}{s}\right)$$

Question 6

Soln:

$$\mathcal{L}^{-1}\left\{\frac{3s-2}{9s^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{3s-2}{9(s^2+25/9)}\right\}$$

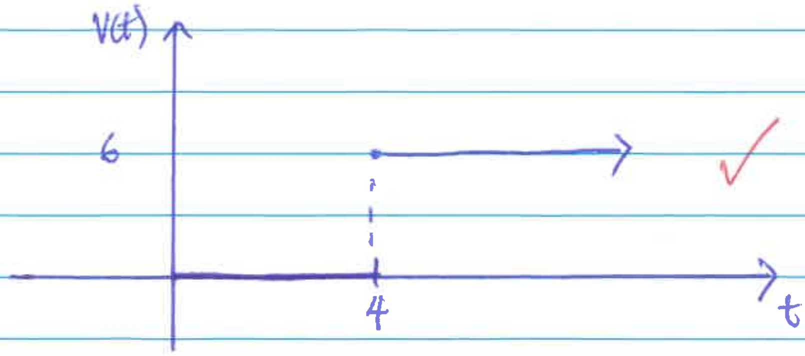
$$= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3s-2}{s^2+25/9}\right\}$$

$$= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{3s}{s^2+25/9} - \frac{3}{5}\cdot\frac{2\cdot 5/3}{(s^2+25/9)}\right\}$$

$$= \frac{1}{9}\left(3\cos\frac{5}{3}t - \frac{6}{5}\sin\frac{5}{3}t\right)$$

Question 7

Soln: $V(t) = 6\,u(t-4)$



Question 8

Soln:
$$\frac{dq}{dt} + q + i_3 = 50\, e^{-t}\, u(t-1)$$
$$\frac{di_3}{dt} + i_3 - q = 0$$

$$s\mathcal{L}\{q\} + \mathcal{L}\{q\} + \mathcal{L}\{i_3\} = \mathcal{L}\{50\, e^{-t}\, u(t-1)\}$$
$$s\mathcal{L}\{i_3\} + \mathcal{L}\{i_3\} - \mathcal{L}\{q\} = 0$$

$$(s+1)\mathcal{L}\{q\} + \mathcal{L}\{i_3\} = 50e^{-1} \cdot \frac{e^{-s}}{s+1} \quad \text{--- (1)}$$
$$-\mathcal{L}\{q\} + (s+1)\mathcal{L}\{i_3\} = 0 \quad \text{--- (2)}$$

$$\begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} \mathcal{L}\{q\} \\ \mathcal{L}\{i_3\} \end{bmatrix} = \begin{bmatrix} 50e^{-1} \cdot \frac{e^{-s}}{s+1} \\ 0 \end{bmatrix}$$

$$\therefore \mathcal{L}\{q\} = \frac{\begin{vmatrix} 50e^{-1} \cdot \frac{e^{-s}}{s+1} & 1 \\ 0 & s+1 \end{vmatrix}}{\begin{vmatrix} s+1 & 1 \\ -1 & s+1 \end{vmatrix}}$$

$$= \frac{50e^{-1} \cdot \frac{e^{-s}}{s+1}(s+1)}{(s+1)^2 + 1}$$

$$= \frac{50\, e^{-1}\, e^{-s}}{(s+1)^2 + 1}$$

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$$\therefore q = \mathcal{L}^{-1}\left\{ 50e^{-1} \cdot \frac{e^{-s}}{(s+1)^2+1} \right\}$$

$$= 50e^{-1}\,\mathcal{L}^{-1}\left\{ \frac{e^{-s}}{(s+1)^2+1} \right\}$$

$$= 50e^{-1}\,\mathcal{L}^{-1}\left\{ e^{-s} \cdot \mathcal{L}\{ e^{-t}\sin t \} \right\}$$

$$= 50e^{-1} \cdot e^{-(t-1)} \sin(t-1)\; u(t-1)$$

b) Recall: $u(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \geq a \end{cases}$

$$\therefore q = 50e^{-1} \cdot e^{-(2-1)} \sin(2-1) \cdot 1$$

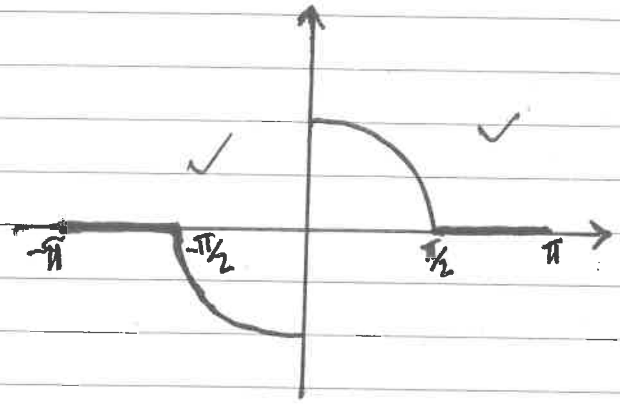
$$= 15.478$$

[10]

Question 9

$$f(x) = \begin{cases} \cos x, & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

(a)



(b)

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \sin nx \, dx + \int_{\pi/2}^{\pi} 0 \cdot \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\pi/2}\left[\sin(n+1)x + \sin(n-1)x\right]dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)x - \frac{1}{n-1}\cos(n-1)x\right]_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\cos(n+1)\frac{\pi}{2} - \frac{1}{n-1}\cos(n-1)\frac{\pi}{2} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\begin{Bmatrix} 0, & n = \text{even} \\ -1, & n = 1, 5, 9, \ldots \\ 1, & n = 3, 7, 11, \ldots \end{Bmatrix} - \frac{1}{n-1}\begin{Bmatrix} 0, & n = \text{even} \\ 1, & n = 1, 5, 9, \ldots \\ -1, & n = 3, 7, 11, \ldots \end{Bmatrix} + \frac{1}{n-1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\begin{cases} \frac{1}{n+1} + \frac{1}{n-1} = \frac{2n}{\pi(n^2-1)}, & n = \text{even} \\ \frac{1}{n+1} - \cancel{\frac{1}{n-1}} + \frac{1}{n+1} + \cancel{\frac{1}{n-1}} = \frac{2}{\pi(n+1)}, & n = 1, 5, 9, \ldots \\ \frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{\pi(n-1)}, & n = 3, 7, 11, \ldots \end{cases}$$

$$f(x) = \frac{1}{\pi}\left[\sin x + \frac{4}{3}\sin 2x + \sin 3x + \ldots\right.$$

Question 10

[6]

$$c_n = \frac{1}{2\pi}\int_0^2 e^{-x}\cdot e^{-in\pi x}\,dx \quad ✓$$

$$= \frac{1}{2\pi}\int_0^2 e^{-(1+in\pi)x}\,dx \quad ✓$$

$$= \frac{1}{2\pi}\cdot\frac{-1}{1+in\pi}\, e^{-(1+in\pi)x}\Big|_0^2 \quad ✓$$

$$= \frac{-1}{2\pi(1+in\pi)}\left[e^{-(1+in\pi)\cdot 2} - e^0\right] ✓$$

$$= \frac{-1}{2\pi(1+in\pi)}\left[e^{-2}\cdot e^{-2in\pi} - 1\right]$$

$$= \frac{-1}{2\pi(1+in\pi)}\left[e^{-2}\left(\overset{1}{\cancel{\cos 2n\pi}} - i\,\overset{0}{\cancel{\sin 2n\pi}}\right) - 1\right] ✓$$

$$= \frac{-1}{2\pi(1+in\pi)}\left(e^{-2}-1\right) \quad ✓$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{-(e^{-2}-1)}{2\pi(1+in\pi)}\, e^{in\pi x}$$

Question 11

$$A(\alpha) = \int_0^1 x\cos\alpha x\,dx + \int_1^\infty 0\cdot\cos\alpha x\,dx \quad ✓$$

let $u = x$ and $dv = \cos\alpha x\,dx$

$du = dx$ $\quad v = \frac{1}{\alpha}\sin\alpha x$

$$\therefore \int x\cos\alpha x\,dx = \frac{x}{\alpha}\sin\alpha x - \int\frac{1}{\alpha}\sin\alpha x\,dx$$

$$= \frac{x}{\alpha}\sin\alpha x + \frac{1}{\alpha^2}\cos\alpha x$$

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$$= \frac{x}{\alpha}\sin\alpha x + \frac{1}{\alpha^2}\cos\alpha x \Big|_0^1$$

$$= \frac{1}{\alpha}\sin\alpha + \frac{1}{\alpha^2}\cos\alpha - \frac{1}{\alpha^2}$$

$$= \frac{\alpha\sin\alpha + \cos\alpha - 1}{\alpha^2}$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha\sin\alpha + \cos\alpha - 1}{\alpha^2}\cos\alpha x\, d\alpha$$