UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS	
MODULE	**MAT2B10 / MAT01B2** **MULTIVARIABLE AND VECTOR CALCULUS**
CAMPUS	**APK**
ASSESSMENT	**EXAMINATION**

DATE 10/11/2020 **TIME 16:30**

ASSESSOR(S) **MR M SIAS**
DR C ROBINSON

INTERNAL MODERATOR **PROF R PANT**

DURATION 180 MINUTES **MARKS 40**

INSTRUCTIONS:

1. WRITE YOUR SOLUTIONS ON SHEETS OF PAPER AND SCAN THEM TO A PDF USING A SCANNING APP.
2. WRITE YOUR SOLUTIONS NEATLY. USE AS FEW PAGES AS POSSIBLE SO THAT THE FILE IS NOT TOO BIG.
3. YOUR SOLUTIONS MUST BE A SINGLE PDF FILE.
4. YOU MUST UPLOAD YOUR SOLUTIONS **BEFORE** 19:30. **NO LATE SUBMISSIONS WILL BE ACCEPTED!**
5. YOU MUST DOWNLOAD YOUR OWN QUESTION PAPER FROM BLACKBOARD. IF YOU ANSWER SOMEONE ELSE'S DOWNLOADED QUESTIONS YOU WILL SCORE ZERO.

6. SHOW ALL CALCULATIONS AND MOTIVATE ALL ANSWERS.
7. NO TEAMWORK OR EXTERNAL ASSISTANCE IS ALLOWED (THIS INCLUDES NON-PROGRAMMABLE CALCULATORS, COMPUTER ALGEBRA SYSTEMS, ANY TYPE OF SOLVERS, AND/OR ANY OTHER TYPE OF MATHEMATICAL SOFTWARE, EITHER OFFLINE OR ONLINE)—IF THERE IS ANY INDICATION THAT YOUR WORK RESEMBLES ANOTHER STUDENT'S WORK, OR IS NOT YOUR OWN WORK, THE CASE WILL BE REFERRED FOR ANALYSIS AND A DISCIPLINARY HEARING AND POSSIBLE EXPULSION FROM THE UNIVERSITY MIGHT FOLLOW.

Question 1 [4 marks]

Let f be a function of two variables that has continuous partial derivatives. Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ are two linearly independent and orthogonal vectors in $\mathbb{R}^2$, that is, $\mathbf{u}_1 \neq c\mathbf{u}_2$ for all $c \in \mathbb{R}$ and $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$. If $D_{\mathbf{u}_1} f(x, y) = k_1$ and $D_{\mathbf{u}_2} f(x, y) = k_2$, show that

$$D_{\mathbf{u}} f(x, y) = (k_1\mathbf{u}_1 + k_2\mathbf{u}_2) \cdot \mathbf{u}.$$

<u>Hint:</u> Use the fact that each $\mathbf{v} \in \mathbb{R}^2$ can be represented uniquely as

$$\mathbf{v} = (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2.$$

Question 2 [5 marks]

Use the method of Lagrange to show that of all the triangles inscribed in a fixed circle, the equilateral one has the largest product of the lengths of the sides.

Question 3 [3 marks]

Set up the triple integral in cylindrical coordinates to find the volume of the solid bounded by the paraboloid $x^2 + y^2 = az$, the xy-plane, and the cylinder $x^2 + y^2 = 2ax$. Take $a > 0$. Do **not** evaluate the integral.

Question 4 [7 marks]

(a) Find $\displaystyle\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\, dV$, where E is the region bounded by the spheres with center the origin and radii r and R, where $0 < r < R$. (4)

(b) For what values of n does the integral in (a) have a limit as $r \to 0^+$. (3)

Question 5 [5 marks]

Use a suitable change of variables to evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y)\, dA,$$

where D is the parallelogram with vertices $(\pi, 0)$, $(0, \pi)$, $(2\pi, \pi)$ and $(\pi, 2\pi)$.

Question 6 [7 marks]

(a) Using Green's Theorem, evaluate the line integral $\oint_C (y - \cos x)\ dx + \sin x\, dy$, where C is the triangle with vertices $(0,0)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, 1\right)$, followed in the anticlockwise direction. (4)

(b) Check your answer in (a) by evaluating the integral $\oint_C (y - \cos x)\, dx + \sin x\, dy$, **directly**. (3)

Question 7 [5 marks]

Consider the vector field $\mathbf{F}$: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by

$$\mathbf{F}(x,y,z) = \langle 3x^2, 2xz - y, z \rangle, \qquad (x,y,z) \in \mathbb{R}^3.$$

(a) Evaluate the following line integrals in the direction of increasing values of t. (4)

(i) $\int_\phi \mathbf{F}(x,y,z) \cdot d\mathbf{r}$, where $\phi(t) = (2t^3, t, t^3)$, $t \in [0,1]$.

(ii) $\int_\psi \mathbf{F}(x,y,z) \cdot d\mathbf{r}$, where $\psi(t) = (2t, t^3, t^2)$, $t \in [0,1]$.

(b) Is $\mathbf{F}(x,y,z)$ a gradient field? Justify your answer using (a). (1)

Question 8 [4 marks]

Consider the vector field $\mathbf{F}$: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by

$$\mathbf{F}(x,y,z) = \langle z^2 - e^y \sin x, e^y \cos x + 2y, 2xz \rangle, \qquad (x,y,z) \in \mathbb{R}^3.$$

(a) Show that $\mathbf{F}(x,y,z)$ is a conservative vector field, and find a potential function for $\mathbf{F}(x,y,z)$. (3)

(b) Evaluate $\int_C (z^2 - e^y \sin x)\, dx + (e^y \cos x + 2y)\, dy + (2xz)\, dz$, where C is the smooth curve from $(0,1,-1)$ to $(\pi, 0, -2)$. (1)