University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MAT2B20 / MAT02B2

LINEAR ALGEBRA 2B

ONLINE SPECIAL EXAM

21 JANUARY 2022

Examiner:	Dr. F. Schulz
Moderator:	Dr. G. Braatvedt
Availability: 08:00 to 11:30	**45** marks

INSTRUCTIONS:

1. The paper consists of **4** pages, **including** the front page.
2. Answer all **ten** questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Read the questions carefully.
5. Questions are to be answered neatly on your own pieces of paper. Please ensure that your details (initials, surname and student number) are clearly written on the first page.
6. Please scan your answers and submit it as a **single** and **neat** pdf document before the deadline which is 11:30.
7. Non-programmable calculators are allowed.
8. This is an open book test.
9. Good luck!

Question 1 [4]

Choose the correct option for each of the multiple choice questions. Submit only the letter corresponding to your final answer for each question by redrawing and completing the table given below.

Question	Answer
1.1	
1.2	
1.3	
1.4	

(1.1) Suppose that A is a 5×8 matrix with nullity$(A^T) = 2$. What is the nullity of A? (1)

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

(1.2) If -2 is an eigenvalue of an invertible matrix A, which value is guaranteed to be an eigenvalue of A^{-1}? (1)

(a) -2 (b) 0 (c) $\frac{1}{2}$ (d) 2 (e) $-\frac{1}{2}$

(1.3) Let $\overline{p} = 1 - x^2$ and $\overline{q} = 4x^2$ be vectors in $\mathcal{P}_2$. Find the cosine of the angle between $\overline{p}$ and $\overline{q}$ when $\mathcal{P}_2$ is equipped with the evaluation inner product at -1, 0 and 1. (1)

(a) $\frac{3}{5}$ (b) $-\frac{2}{\sqrt{5}}$ (c) $-\frac{3}{5}$ (d) 1 (e) 0

(1.4) Suppose that the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \text{ and } T\left(\begin{bmatrix}-1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix}.$$

Which one of the matrices below is the standard matrix for T? (1)

(a) $\begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$ (c) $\begin{bmatrix}1 & -2\\1 & -1\end{bmatrix}$ (d) $\begin{bmatrix}1 & -1\\1 & -2\end{bmatrix}$ (e) None of these

Question 2 [4]

Determine whether the following statements are TRUE or FALSE. Motivate the statement if TRUE; provide a counter-example if FALSE.

(2.1) If A is a square matrix with $\det(A) = \pm 1$, then A is an orthogonal matrix. (2)

(2.2) If $S : V \to W$ and $T : V \to W$ are linear transformations and α is a scalar, then the transformation $R : V \to W$ defined by $R(\overline{x}) = \alpha T(\overline{x}) + S(\overline{x})$ for $\overline{x} \in V$ is also linear. (2)

Question 3 [3]

Give an example (including details) of the following or briefly justify why such an example does not exist.

(3.1) A 3×3 orthogonally diagonalizable matrix which does not have 3 distinct eigenvalues. (1)

(3.2) A linear system of equations which does not have a least squares solution. (1)

(3.3) Two onto operators T and S on a vector space V such that $T \circ S$ is not onto. (1)

Question 4 [5]

Suppose that the matrix A has a characteristic polynomial given by

$$p(\lambda) = (\lambda - 6)^3(\lambda + 2)^5(\lambda + 1).$$

(4.1) What is the largest dimension that the eigenspace corresponding to -2 may have? (1)

(4.2) Is $-I - A$ invertible? (1)

(4.3) What is the rank of A? (1)

(4.4) Is it possible for A to be diagonalizable? And if so, what geometric multiplicities will each of the eigenvalues have in this case? (2)

Question 5 [8]

Let $A = \begin{bmatrix} a & b & -2 \\ b & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, where a and b are real numbers.

(5.1) Exactly one of the following vectors is not an eigenvector of A, the other two are eigenvectors of A corresponding to *distinct* eigenvalues. Determine the vector that is not an eigenvector of A. (2)

$$(1, 0, -1), \quad (0, 1, 1), \quad (0, 1, 0).$$

(5.2) Hence, determine the values of a and b and find the eigenvalues and a corresponding eigenvector for each eigenvalue of A. (4)

(5.3) Is A orthogonally diagonalizable? If so, find matrices P and D such that $A = PDP^T$. If not, explain why this is not the case. (2)

Question 6 [3]

For vectors $\overline{p}, \overline{q} \in \mathcal{P}_2$ we define the following operation:

$$\langle \overline{p}, \overline{q} \rangle = p'(0)q'(0) + p'(1)q'(1) + p'(2)q'(2),$$

where p' and q' are the derivatives of p and q, respectively. Does this operation define an inner product on $\mathcal{P}_2$? Motivate your answer clearly.

Question 7 [3]

Let V be a finite-dimensional inner product space. It is known that if W is a *subspace* of V, then $(W^\perp)^\perp = W$. Prove or disprove: If W is any nonempty *subset* of V, then it still holds that $(W^\perp)^\perp = W$.

Question 8 [4]

Consider the quadratic form $-2x^2 - 4y^2 - 2xy = 1$.

(8.1) Determine whether the quadratic form is positive definite, negative definite, or indefinite. Show all calculations. (3)

(8.2) Hence, is this conic section a hyperbola, ellipse or neither? Explain. (1)

Question 9 [5]

(9.1) Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be the linear operator defined by

$$T(a + bx + cx^2 + dx^3) = (a - c)x - ax^3.$$

Find a basis for $\ker(T)$ and $R(T)$ and use it to decide whether or not $\ker(T) = R(T)$? (3)

(9.2) Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. If $\dim(V)$ is odd, prove that $\ker(T) \neq R(T)$. (2)

Question 10 [6]

Let $S = \{1, x, x^2\}$ be the standard basis for $\mathcal{P}_2$ and consider the basis

$$C = \{(1, -1, 0), (1, 0, 0), (0, 0, 1)\}$$

for $\mathbb{R}^3$. Suppose that the linear transformation $T : \mathcal{P}_2 \to \mathbb{R}^3$ satisfies

$$T(1) = (1, -1, 0), \;\; T(x) = (1, 0, 0) \text{ and } T(x^2) = (0, 0, 1).$$

(10.1) Prove that T^{-1} exists and find its domain and codomain. (3)

(10.2) If $B = \{\overline{p}_1, \overline{p}_2, \overline{p}_3\}$ is a basis for $\mathcal{P}_2$ and

$$[T^{-1}]_{B,C} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

find $\overline{p}_1$, $\overline{p}_2$ and $\overline{p}_3$. (3)