University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE	**MAT3A20/MAT02A3** DISCRETE MATHEMATICS AND LOGIC 3A
CAMPUS	**APK**
SUPPLEMENTARY EXAM	**JULY 2021**

Examiner	Dr F Schulz
Moderator	Prof W Conradie (WITS)
Duration 180 Minutes	65 Marks

Surname and initials:____________________

Student number:____________________

Identity number: ____________________

INSTRUCTIONS:

1. The paper consists of **11** printed pages, **excluding** the front page.
2. Read the questions carefully and answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
5. Non-programmable calculators are allowed.

Question 1 [4]

Let $A = \{a, b, c, e, f, h, i\}$ and $B = \{\varnothing, b, c, g, h\}$.

(a) $A \cap B =$____ (1)

(b) How many elements are there in the set $\mathscr{P}(A) \times \mathscr{P}(B)$? (1)

(c) What is the largest possible length of an Eulerian circuit in a graph G with $V(G) = A$? Justify your answer. (2)

Question 2 [3]

Let $f : \mathbb{N} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by the formulas

$$f(x) = x + 1 \text{ and } g(x) = \left((x-1)^2, |x-1|\right),$$

respectively. Explain why $(gf)^{-1}$ exists but does not equal $f^{-1}g^{-1}$.

Question 3 [5]

(a) Show that if $\mathcal{P}$ is a partition of a set X, then the relation $\sim_{\mathcal{P}} \subseteq X \times X$, defined by $x \sim_{\mathcal{P}} y$ if and only if x and y belong to the same member of $\mathcal{P}$, is an equivalence relation on X. (3)

(b) Can we find an equivalence relation R on $\mathbb{Z}$, and integers x and y such that $R[\{x\}] \neq R[\{y\}]$ and $R[\{x\}] \cap R[\{y\}] \neq \varnothing$? If so, give an example of such an equivalence relation. If not, explain why not. (2)

Question 4 [3]

Suppose A, B, C and D are propositional formulas. If $(A \vee B) \to \neg(C \wedge D)$ is false and $\neg C \leftrightarrow (B \wedge D)$ is true, what is the truth value of A? Explain.

Question 5 [4]

Negate the following propositional formula and use known logical equivalences to transform the result into an equivalent formula in negation normal form.

$$(p \leftrightarrow q) \vee (p \wedge \neg r)$$

Question 6 [6]

Consider the following argument:

Sherlock is clever or he cannot solve the case, but not both.
John assisting on the case is necessary for Sherlock to solve the case.

John assists on the case whenever Sherlock is not clever.

(a) Write down the inference rule on which the above argument is based. (2)

(b) Determine whether it is logically valid using semantic tableaux. If not, provide a falsifying valuation. (4)

Question 7 [7]

In my garden there are 34 animals. 19 of these animals can fly, 12 of them are mammals, and 3 are rodents. Recall that all rodents are mammals and that no rodent can fly. 8 of the animals are not mammals and also cannot fly.

(a) How many of the animals have at least one of these properties? (In other words, how many can fly or are mammals or are rodents?) (2)

(b) How many of the mammals can fly? (3)

(c) How many of the animals are non-flying mammals? (2)

Question 8 [4]

Let S be a set of ten integers chosen from 1 through 50.

(a) How many different subsets of four integers does S have? (1)

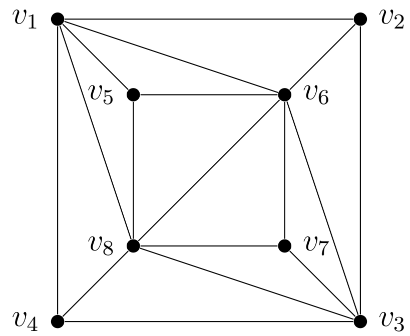
(b) Show that S contains at least two different (but not necessarily disjoint) subsets of four integers that add up to the same number. (3)

Question 9 [5]

Consider the recurrence relation $a_n = 4a_{n-1} + (-1)^n$ for $n > 0$ with initial condition $a_0 = 1$. Solve the recurrence relation by making use of generating functions.

Question 10 [8]

Consider the graph G depicted below and answer the questions that follow:



(a) Use the First Theorem of Graph Theory to find $m(G)$. (2)

(b) Does $\langle v_1, v_2, v_4, v_5, v_7 \rangle_G$ have a spanning tree? If so, draw and label such a spanning tree; if not, motivate your answer clearly by referring to the definition of a spanning tree. (2)

(c) Is G Hamiltonian? Motivate your answer. (2)

(d) Determine $\chi(G)$. Motivate your answer clearly. (2)

Question 11 [5]

(a) Prove that a graph G is a tree if and only if there exists a unique $u - v$ path in G for any two vertices $u, v \in V(G)$. (3)

(b) Does the existence of two distinct $u - v$ paths in a graph G imply that there is a cycle in G containing u and v. Motivate your answer by providing either a short proof or a counter-example. (2)

Question 12 [5]

Electronic components A to H need to be connected to each other as indicated in the following table, ✓ indicating the presence of a connection and × indicating the absence of a connection:

	B	C	D	E	F	G	H
A	✓	×	×	✓	×	×	✓
B		✓	×	×	×	✓	×
C			✓	×	✓	×	×
D				✓	×	×	✓
E					✓	×	×
F						✓	×
G							✓

Model this information as a graph and use it to determine whether these components in this configuration can be embedded in a single layer on a circuit board without any connections crossing.

Question 13 [7]

(a) Below is an incomplete proof that every planar graph is 5-colourable. Complete the entire proof by providing, in full detail, the missing part at the end indicated by '...'. (5)

Let G be any connected plane graph. We proceed by induction on the number n of vertices. If $n \leq 5$ the graph is obviously 5-colourable. So assume that all planar graphs with fewer than n vertices, $n > 5$, are 5-colourable, and suppose that G is a plane graph with n vertices. By a previous theorem there is a vertex v in G such that $\deg v \leq 5$. We delete v from G to obtain the plane graph $G - v$. By the induction hypothesis $G - v$ is 5-colourable. Let a 5-colouring of $G - v$ with colours 1,2,3,4, and 5 be given. Excluding a few trivial cases, we may assume, without loss of generality, that the vertices adjacent to v in G are v_1, v_2, v_3, v_4 and v_5, and that v_i is coloured with colour i, $1 \leq i \leq 5$. We may also assume that v_1, v_2, v_3, v_4 and v_5 are arranged cyclically around v, in that order. Let H be the subragph of $G - v$ induced by all vertices coloured either 1 or 3. If v_1 and v_3 belong to different components of H, then it can be shown that G has a 5-colouring. So suppose that v_1 and v_3 belong to the same component of H. ...

(b) Is the converse of the 5-colour theorem above true? Motivate your answer by providing either a short proof or a counter-example. (2)