1 Exam [35]

Instructions:
Please answer the questions carefully, and clearly write your student number and name.
When uploading your exam, upload the document as a pdf.
When saving your document, please save the file with your student number then name (such as "201912345-JohnPaul.pdf").
Note that the deadline to submit the exam is 11:30 on Monday 7 June 2021.
Additionally be aware that there are 9 questions and 2 pages of this exam.

Question 1 [8]

Determine whether the following statements are true or false. If true, give a short justification. If false, give a counter example.

(a) A symmetric matrix is either positive definite, negative definite, or indefinite.

(b) If $\overline{v}$ is a nonzero vector in V, then there is exactly one linear transformation $T : V \to W$ such that $T(-\overline{v}) = -T(\overline{v})$.

(c) Every linear transformation from M_{33} to $\mathcal{P}_9$ is an isomorphism.

(d) The composition of two linear transformations is also a linear transformation.

Question 2 [4]

Let A be a $m \times n$ matrix. Suppose that the nullspace of A is a plane in $\mathbb{R}^3$ and the range of T_A is spanned by a nonzero vector $\overline{v}$ in $\mathbb{R}^5$. Find m and n as well as the rank and nullity of A.

Question 3 [5]

Let A be a 3×3 matrix with eigenvalues 2 and 1. If the eigenspace corresponding to 2 has basis $\{(-1, 1, 0), (0, 0, 1)\}$ and the eigenspace corresponding to 1 has basis $\{(0, -1, 1)\}$, find A^{25}.

Question 4 [3]

Let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

Prove, or disprove, that $\langle \overline{x}, \overline{y} \rangle = \overline{x}^T A \overline{y}$ is an inner product. Note that $\overline{x}, \overline{y} \in \mathbb{R}^2$.

Question 5 [3]

Find the best approximation to

$$\overline{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

from the column space of

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Question 6 [4]

Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form

$$Q = 4x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2 - 4x_2x_3,$$

and express Q in terms of the new variables.

Question 7 [2]

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator such that

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and } T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Solve for $T\left(\begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix}\right)$.

Question 8 [2]

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by the formula $T(x, y) = (x - 2y, -y)$. Is T one-to-one? If so, find T^{-1}.

Question 9 [4]

Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 5 \\ 0 & -2 & 6 \end{bmatrix}$$

be the matrix for $T : \mathcal{P}_2 \to \mathcal{P}_2$ with respect to the basis $B = \{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$, where $\overline{v}_1 = 2 - 2x^2$, $\overline{v}_2 = 1 + 2x + 2x^2$, and $\overline{v}_3 = 3x + x^2$.

(a) Find $[T(\overline{v}_1)]_B$, $[T(\overline{v}_2)]_B$, and $[T(\overline{v}_3)]_B$.

(b) Find $T(\overline{v}_1)$, $T(\overline{v}_2)$, and $T(\overline{v}_3)$.

(c) Find $T(1 + x^2)$.