Question Number	Mark Award
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APPLIED MATHEMATICS

Introduction to Numerical analysis
APM02B2/APM2B10

Supplementary Examination: 01/12/2021

Duration: 3 hours + 30 minutes admin **Marks:** 100
Assessor: Dr F. Chirove and Dr J Homann
Moderator: Prof E. Momoniat

Surname: ____________________ **Assigned Number:** ______________

Instructions:

1. Check that this question paper consists of 3 pages in total.
2. **Carefully read and follow the instructions of each question.**
3. Calculators are permitted.
4. If you answer a question paper that you did not download yourself then you get ZERO mark automatically.
5. Answer the questions in order (from 1 to 5) and write out your solutions on sheets of paper. Cross out answers that are not to be marked. **Your assigned number, surname and student number must be written at the top of each page.**
6. All calculations must be shown.
7. Use a scanning app (CamScanner is a good option) to scan your solutions into a PDF. Your solutions must be one PDF. Pages must be oriented correctly, i.e. not upside down or on their sides. Do not upload JPEG files.
8. Check that your PDF is not too big (it should be around 1MB/page). SAVE YOUR PDF AS "assignednumber-surname" eg 123-Mango
9. To submit - Scroll down to "ASSIGNMENT SUBMISSION" and then "Attach files". Select your PDF and click "Submit".
10. FOR ANY ISSUES, EMAIL apm02a2@uj.ac.za

Question 1 (20 marks)

(a) Let $f\colon I \to \mathbb{R}$ not be continuous, where I is a closed interval in $\mathbb{R}$. With appropriate reasoning, is the Bisection Method applicable to f? (2)

(b) Given $f(x) = xe^x - \cos x$ on $[0, 1]$, with reference to the Bisection Method, determine the minimum number of iterations required to have an accuracy of 10^{-9} in the root. If no root exists on the interval, then write "No root exists." (2)

(c) Given $f(x) = xe^x - \cos x$, use the Bisection Method to approximate a root of f on $[0, 1]$, accurate to 3 decimal digits. If no root exists on the interval, then write "No root exists." (8)

(d) Apply the fixed point iteration to $-\frac{1}{2}\sin x + x = 1$ four times, with $x_0 = \frac{1}{2}$. Use an accuracy of 6 decimal digits throughout. (4)

(e) Determine if the fixed point iteration applied in the previous question will converge. (4)

Solution

a Since the function is not continuous, the Bisection Method is not applicable as a jump discontinuity may occur at 0. For example, let $I = [-1, 1]$ and let $f\colon I \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{whenever } x < 0, \\ 1 & \text{whenever } x = 0, \\ x & \text{whenever } x > 0. \end{cases}$$

It is trivial to show that f is not continuous at 0. We have $f(-1) f(1) < 0$, but f has no root.

b Since f is continuous on $I := [0, 1]$ and $f(0) f(1) < 0$, f has at least one root on I and the Bisection Method is applicable. For root tolerance, given the closed interval I and a tolerance $0 < 10^{-9} =: \epsilon \in \mathbb{R}$, with x_0 the exact root and x_n the approximate root after n iterations, we require

$$\begin{aligned} |x_n - x_0| \leq \frac{|I|}{2^n} &< \epsilon \\ &\Rightarrow n > \log_2\left(\frac{|I|}{\epsilon}\right) \\ &\Rightarrow n > \log_2\left(\frac{1}{10^{-9}}\right) \approx 29.897 \\ &\Rightarrow n \geq 30, \end{aligned}$$

so the minimum is $n = 30$.

c The tolerance is specified to be 3 decimal digits, so $\epsilon = 10^{-2}$. The Bisection Method is now applied. By letting n denote the number of iterations, the following table is obtained after applying the Bisection Method to the given function, on the given interval, with a tolerance of $\epsilon = 0.001$, a maximum number of iterations allowed of 50 and by rounding every presented value (unless otherwise stated), to 6 decimal digits (where full values are used for calculations), the trailing zeroes are dropped, the following data can be presented, where we choose the interval after an iteration to have left endpoint a and right endpoint c if the product of the function values evaluated at a and c is strictly negative and left endpoint c and right endpoint b if the product of the function values evaluated at c and b is strictly negative.

n	I_n	c	$f(c)$	Is $f(c) < \epsilon$?
1	$[0, 1]$	0.5	-0.053222	No.
2	$[0.5, 1]$	0.75	0.856061	No.
3	$[0.5, 0.75]$	0.625	0.356691	No.
4	$[0.5, 0.625]$	0.5625	0.141294	No.
5	$[0.5, 0.5625]$	0.53125	0.041512	No.
6	$[0.5, 0.53125]$	0.515625	-0.006475	No.
7	$[0.515625, 0.53125]$	0.523438	0.017362	No.
8	$[0.515625, 0.523438]$	0.519531	0.005404	No.
9	$[0.515625, 0.519531]$	0.517578	-0.000545	Yes.

The process terminated after 9 iterations by satisfying the tolerance condition. An approximation of a root of f on I is $c = 0.517578$.

d Let $f(x) := -\frac{1}{2}\sin x - 1$, then f facilitates fixed point iteration of $-\frac{1}{2}\sin x + x = 1$. Let $x_0 := \frac{1}{2}$ denote the initial value, then $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}_0$. The values presented, as instructed, as rounded to 6 decimal digits and rounded values, as instructed, are used for calculations (trailing zeroes are dropped).

n	x_n	$x_{n+1} = f(x_n)$
0	0.5	-1.239713
1	-1.239713	-0.527155
2	-0.527155	-0.748462
3	-0.748462	-0.659744

Hence, $x_4 = -0.659744$.

e Let $f(x) := -\frac{1}{2}\sin x - 1$, then $f'(x) = -\frac{1}{2}\cos x$, so $f'\left(\frac{1}{2}\right) = -0.239713 \Rightarrow \left|f'\left(\frac{1}{2}\right)\right| = 0.239713 < 1 \Rightarrow$ the process will converge to the root near x_0.

Question 2 (20 marks)

(a) Consider the task of approximating $\displaystyle\int_1^2 e^{3x^2-x}dx$ using the Composite Trapezoidal rule. How large should n and h be chosen in order to ensure that the error is at most 0.0001? (9)

Solution

$f(x) = e^{3x^2-x}$, $f'(x) = e^{3x^2-x}(6x-1)$, $f''(x) = e^{x(3x-1)}\left(36x^2 - 12x + 7\right)$, $f'''(x) = e^{x(3x-1)}\left(216x^3 - 108x^2 + 126x - 19\right)$

$f'''(x) = 0$ when $x = 1/6 \notin [1, 2]$

$M = max\{|f''(1)|, |f''(2)|\} = max\{229.061, 2.79736 \times 10^6\} = 2.79736 \times 10^6$

$\dfrac{(b-a)Mh^2}{12} \leq 0.0001 \implies h^2 \leq 5.44799 \times 10^{-8}$ and $h \leq 0.000233409$.

hence

$n \geq 4284.32 = 4285$
$h = 1/4285 = 0.000233372$

(b) The composite trapezoidal rule is used to approximate the integral

$$I = \int_1^2 x(x + x^3)dx.$$

i. Complete the table below leaving all your solutions correct to four decimal places. (4)

x_i	1	1.2	1.25	1.4	1.5	1.6	1.75	1.8	2
$f(x_i)$	2	3.8016	4.00391	5.8016	7.3125	9.1136	12.4414	13.7376	20

ii. Using the data in the table in part (i), apply the composite trapezoidal rule to find the approximate value of I on $[1, 2]$ with $h = 0.25$. (3)

Solution

$I = \dfrac{0.25}{2}(2 + 20 + 2(4.00391 + 7.3125 + 12.4414)) = 8.6895$

(c) Find the approximate value of the integral $I = \int_0^{\pi/4} e^{3x}\sin(2x)\, dx$ using Gaussian quadrature with $n = 2$, the nodes $t_1 = -0.577,\ t_2 = 0.577$ and coefficients $c_1 = c_2 = 1$. All your solutions should be expressed correct to three decimal points. (4)

Solution:

Transforming the given integral using $t = \dfrac{2x - \pi/4 - 0}{\pi/4 - 0}$ *or* $x = \dfrac{\pi}{8}(t + 1)$ *we get*

$\int_0^{\pi/4} e^{3x}\sin(2x)\, dx = \dfrac{\pi}{8}\int_{-1}^{1} e^{\frac{3\pi}{8}(t+1)} \sin\left(\dfrac{2\pi}{8}(t + 1)\right) dt$

Using the given nodes and coefficients for $n = 2$

$f(t_1) = 0.210812,\ f(t_2) = 2.37956$

$\int_0^{\pi/4} e^{3x}\sin(2x)\, dx \approx= c_1 f(t_1) + c_2 f(t_2) = 2.590374$

Question 3 (20 marks)

Let $f(x) = -x\cos(2x) + x^2, \quad x_0 = 0, \quad x_1 = 0.3, \quad x_2 = 0.7.$

(a) Find Lagrange interpolating polynomial for $f(x)$ using the three given nodes. (4)

Solution:

$f(0) = 0, f(0.3) = -0.15760; f(0.7) = 0.37102$

$L_0(x) = \frac{(x-0.3)(x-0.7)}{(0-0.3)(0-0.7)} = 4.76190(x - 0.7)(x - 0.3)$

$L_1(x) = \frac{(x-0)(x-0.7)}{(0.3-0)(0.3-0.7)} = -8.33333(x - 0.7)x$

$L_2(x) = \frac{(x-0)(x-0.3)}{(0.7-0)(0.7-0.3)} = 3.57143(x - 0.3)x$

$P_2(x) = f(0)L_0(x) + f(0.3)L_1(x) + f(0.7)L_2(x)$

$= 1.31334(x - 0.7)x + 1.32508(x - 0.3)x = 2.63842x^2 - 1.31686x$

(b) Using the nodes x_0 and x_1, construct the Hermite interpolating polynomial $H_3(x)$ for $f(x)$ using the Lagrange coefficient polynomials. (5)

Solution:

$f(0) = 0, f(0.3) = -0.15760$
$f'(x) = 2x + 2x\sin(2x) - \cos(2x), f'(0) = -1, f'(0.3) = 0.11345$
$L_0(x) = \frac{(x-0.3)}{(0-0.3)} = -3.33333(x - 0.3)$
$L_1(x) = \frac{(x-0)}{(0.3-0)} = 3.33333x$
$L_0'(x) = -3.33333, \quad L_1'(x) = 3.33333$
$H_0(x) = [1 - 2(x - 0)L_0'(0)]L_0^2 = 11.1111(x - 0.3)^2(6.66667x + 1)$
$H_1(x) = [1 - 2(x - 0.3)L_1'(0)]L_1^2 = 11.1111(1 - 6.66667(x - 0.3))x^2$
$\hat{H}_0(x) = (x - 0)L_0^2 = 11.1111(x - 0.3)^2x$
$\hat{H}_1(x) = (x - 0.3)L_1^2 = 11.1111(x - 0.3)x^2$
$H_3(x) = f(0)H_0(x) + f(0.3)H_1(x) + f'(0)\hat{H}_0(x) + f'(0.3)\hat{H}_1(x)$
$= -37.037x^2(x - 0.3) - 1.75112(1 - 6.66667(x - 0.3))x^2 - 37.037x(x - 0.3)^2$
$= -62.3999x^3 + 28.08x^2 - 3.33333x$

(c) Determine the natural cubic spline that interpolates the data (7)

$$f(0) = 2,\ f(3) = 3,\ f(8) = 1$$

and find the approximate value of $f(1.2)$.

Solution:

$$\begin{aligned} S_0(x) &= a_0 + b_0 x + c_0 x^2 + d_0 x^3, \\ S_1(x) &= a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3, \\ h_0 &= 3, h_1 = 5, a_0 = 2, a_1 = 3, a_2 = 1, c_0 = c_2 = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 16 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ \frac{3(a_2-a_1)}{h_1} - \frac{3(a_1-a_0)}{h_0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{11}{5} \\ 0 \end{pmatrix} \\ \mathbf{c} &= \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

So from $\mathbf{Ax} = \mathbf{b}$, we have $16c_1 = -\frac{11}{5}$ implying $c_1 = -(11/80) = -0.1375$; and the rest of the parameters are given in the table below

a_0	b_0	c_0	d_0	a_1	b_1	c_1	d_1
2	0.470833	0	−0.0152778	3	0.0583333	−0.1375	0.00916667

$$\begin{aligned} S_0(x) &= -0.0152778x^3 + 0.0583333x + 2 \ on\ [0,3], \\ S_1(x) &= 0.00916667(x-3)^3 - 0.1375(x-3)^2 + 0.0583333(x-3) + 3,\ on\ [3,8], \\ &= -0.0166667x^3 + 0.4x^2 - 3.18333x + 9.4,\ on\ [3,8]. \end{aligned}$$

$$S(x) = \begin{cases} S_0(x), on\ [0,3] \\ S_1(x), on\ [3,8] \end{cases} \qquad (2)$$

$f(1.2) \approx S(1.2) = S_0(1.2) = -0.0152778(1.2)^3 + 0.0583333(1.2) + 2 = 2.0436$

(d) The cubic Legendre polynomial is $P_2(x) = \dfrac{1}{2}(5x^3 - 3x)$. Prove that it is orthogonal (over $[-1,1]$) to all polynomials of degree 2. (4)

Solution:

Let the general polynomial of degree 2 be given by $L_2(x) = ax^2 + bx + c$,

Then

$\int_{-1}^{1} P_2(x)L_2(x)dx = \int_{-1}^{1} \dfrac{1}{2}(5x^3 - 3x)(ax^2 + bx + c)dx$

$= \displaystyle\int_{-1}^{1} \left(\frac{5ax^5}{2} - \frac{3ax^3}{2} + \frac{5bx^4}{2} - \frac{3bx^2}{2} + \frac{5cx^3}{2} - \frac{3cx}{2} \right) dx$

$$
\begin{aligned}
&= \left[\frac{1}{2}\left(\frac{1}{4}x^4(5c-3a)+\frac{5ax^6}{6}+bx^5-bx^3-\frac{3cx^2}{2}\right)\right]_{-1}^{1} \\
&= \left[\frac{1}{2}\left(\frac{1}{4}(5c-3a)+\frac{5a}{6}+b-b-\frac{3c}{2}\right)\right]-\left[\frac{1}{2}\left(\frac{1}{4}(5c-3a)+\frac{5a}{6}-b+b-\frac{3c}{2}\right)\right] \\
&= 0
\end{aligned}
$$

Therefore $P_2(x)$ is orthogonal to all polynomials of order 2.

Question 4 (20 marks)

(a) Develop a first-order method for approximating $f''(x)$ which uses the data $f(x-4h)$, $f(x+3h)$, $f(x)$ and $f(x+5h)$. (13)

(b) Using the most accurate centered difference formula, approximate $f''(1)$ and then state the error in the approximation, given $f(x)=\cos(x)+x$, for each $h \in \{0.1, 0.01, 0.001\}$. Use an accuracy of 6 digits throughout. (7)

Solution

a Let $\xi_1 \in (x-4h, x)$, $\xi_2 \in (x, x+3h)$ and $\xi_3 \in (x, x+5h)$, then

$$f(x-4h) = f(x) - 4hf'(x) + \frac{16}{2}h^2 f''(x) - \frac{64}{6}h^3 f'''(\xi_1), \tag{3a}$$

$$f(x+3h) = f(x) + 3hf'(x) + \frac{9}{2}h^2 f''(x) + \frac{27}{6}h^3 f'''(\xi_2) \quad \text{and} \tag{3b}$$

$$f(x+5h) = f(x) + 5hf'(x) + \frac{25}{2}h^2 f''(x) + \frac{125}{6}h^3 f'''(\xi_3). \tag{3c}$$

By performing 2(3a)+(3b)+(3c) and simplifying, we obtain the following.

$$
\begin{aligned}
f''(x) &= \frac{2f(x-4h)+f(x+3h)+f(x+3h)-4f(x)}{33h^2} \\
&+ \frac{h}{198}\left[128f'''(\xi_1) - 27f'''(\xi_2) - 125f'''(\xi_3)\right].
\end{aligned}
$$

b The required formula is

$$f''(x) = \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2} + \mathcal{O}\left(h^4\right).$$

Given $f(x) = x + \cos x$, we have $f''_{\text{exact}}(x) = -\cos x$ and so $f''_{\text{exact}}(1) = -0.540302$. By letting $D(h,1)$ denote the numerical approximation of the second derivative at 1, using a step-size h, and letting $E(h,1) := |D(h,1) - f''_{\text{exact}}(1)|$, then the following table is populated.

h	$D(h,1)$	$E(h,1)$
0.1	-0.540302	0.0
0.01	-0.540302	0.0
0.001	-0.540302	0.0

Question 5 (20 marks)

(a) Given the initial value problem (10)

$$x + y + \sin(x+y) + \cos(xy) + \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad y(0) = 0,$$

approximate $y(1)$, with $h = 0.1$.

Solution:

n	x_n	y_n
0	0.0	0.0
1	0.1	-0.1
2	0.2	-0.199995
3	0.3	-0.299916
4	0.4	-0.399528
5	0.5	-0.498348
6	0.6	-0.59559
7	0.7	-0.615024
8	0.8	-0.690155
9	0.9	-0.780679
10	1.0	-0.865666

(b) The Runge-Kutta method of order 2 (RK2) with $h = 0.1$ is used to solve (10)

$$\frac{dy}{dx} = -y + xy$$

with $y(0) = 1$ in order to find $y(0.3)$ correct to four decimal places. Assuming that the local error in RK2 is given by

$$\epsilon_{i+1} = \frac{h^3}{6} y'''(\xi),\ \xi \in [x_i, x_{I+1}],$$

estimate an upper bound for the global error at $x = 0.3$.

Solution:

$\Delta_3 = \epsilon_3 + \alpha_2\epsilon_2 + \alpha_2\alpha_1\epsilon_1$ so that

$|\Delta_3| \leq \max\limits_{[0,0.3]} |\epsilon_m|(1+\alpha+\alpha^2) = \max\limits_{[0,0.3]} |\epsilon_m| \left(\dfrac{\alpha^3-1}{\alpha-1}\right)$ where

$\alpha = \max\limits_{[0,0.3]} |\alpha_m| = 1 + h \max\limits_{[0,0.3]} |F_y|, \quad h = 0.1.$

$\max\limits_{[0,0.3]} |\epsilon_m| = \max\limits_{[0,0.3]} \left|\dfrac{h^3}{6} y'''\right| = \max\limits_{[0,0.3]} \left|\dfrac{h^3}{6}(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2)\right|$

$f_x = y, f_y = -1 + x, f_{xx} = f_{yy} = 0, \quad f_{xy} = 1$

$$f_{xx}+2ff_{xy}+f^2f_{yy}+f_xf_y+ff_y^2 = (x-1)^2(xy-y)+(x-1)y+2(xy-y) = x^3y-3x^2y+6xy-4y$$

$F(x,y) = \frac{1}{2}(-h(xy-y) + (h+x)(h(xy-y)+y) - y) + \frac{1}{2}(xy-y)$

$= \frac{1}{2}h^2xy - \frac{h^2y}{2} + \frac{1}{2}hx^2y - hxy + hy + xy - y$

$F_y = \frac{h^2x}{2} - \frac{h^2}{2} + \frac{hx^2}{2} - hx + h + x - 1$

i	0	1	2	3
x_i	0	0.1	0.2	0.3
$f(x_i)$	1	0.9095	0.835467	0.775146

From RK2

$$\alpha = 1 + h \max_{[0,0.3]} |F_y| = 1 + 0.1(0.0905) = 1.0905$$

$$\max_{[0,0.3]} |\epsilon_m| = \max_{[0,0.3]} \left| \frac{h^3}{6}(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2) \right| = \frac{h^3}{6}(1.89368) = 0.000315614$$

$$|\Delta_3| \leq \max_{[0,0.3]} |\epsilon_m| \left(\frac{\alpha^3 - 1}{\alpha - 1} \right) = (0.000315614) \left(\frac{1.0905^3 - 1}{1.0905 - 1} \right) = 0.00103511$$

END OF QUESTION PAPER