Question Number	Mark Award
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APPLIED MATHEMATICS

Introduction to Numerical analysis
APM02B2/APM2B10

Special Examination: 19/01/2022

Duration: 3 hours + 30 minutes admin **Marks:** 100
Assessor: Dr F. Chirove and Dr J Homann
Moderator: Prof E. Momoniat

Surname: ________________ **Assigned Number:** ____________

Instructions:

1. Check that this question paper consists of 3 pages in total.
2. **Carefully read and follow the instructions of each question.**
3. Calculators are permitted.
4. If you answer a question paper that you did not download yourself then you get ZERO mark automatically.
5. Answer the questions in order (from 1 to 5) and write out your solutions on sheets of paper. Cross out answers that are not to be marked. **Your assigned number, surname and student number must be written at the top of each page.**
6. All calculations must be shown.
7. Use a scanning app (CamScanner is a good option) to scan your solutions into a PDF. Your solutions must be one PDF. Pages must be oriented correctly, i.e. not upside down or on their sides. Do not upload JPEG files.
8. Check that your PDF is not too big (it should be around 1MB/page). SAVE YOUR PDF AS "assignednumber-surname" eg 123-Mango
9. To submit - Scroll down to "ASSIGNMENT SUBMISSION" and then "Attach files". Select your PDF and click "Submit".
10. FOR ANY ISSUES, EMAIL apm02a2@uj.ac.za

Question 1 (20 marks)

(a) Let $f\colon I \to \mathbb{R}$ be continuous, where I is a closed interval in $\mathbb{R}$. State the conditions required to apply the Bisection Method to f. (2)

(b) Given $f(x) = 2xe^x - 2\cos x$ on $[0, 1]$, with reference to the Bisection Method, determine the minimum number of iterations required to have an accuracy of 10^{-9} in the root. If no root exists on the interval, then write "No root exists." (2)

(c) Given $f(x) = 2xe^x - 2\cos x$, use the Bisection Method to approximate a root of f on $[0, 1]$, accurate to 3 decimal digits. If no root exists on the interval, then write "No root exists." (8)

(d) Apply the fixed point iteration to $-\sin x + 2x = 2$ four times, with $x_0 = \frac{1}{2}$. Use an accuracy of 6 decimal digits throughout. (4)

(e) Determine if the fixed point iteration applied in the previous question will converge. (4)

Question 2 (20 marks)

(a) Consider the task of approximating $\displaystyle\int_0^1 e^{-4x^2}\,dx$ using the Composite Trapezoidal rule. How large should n and h be chosen in order to ensure that the error is at most 0.001? (9)

(b) The composite trapezoidal rule is applied with $h = 0.2$ to approximate the integral

$$I = \int_0^1 x(1 + x^2)dx.$$

i. Complete the table below leaving your solutions correct to four decimal places. (3)

x_i	0	0.2	0.4	0.6	0.8	1
$f(x_i)$	A	B	C	D	E	F

ii. Find the approximate value of I on $[0, 1]$ using the composite trapezoidal rule. (3)

(c) Find the approximate value of the integral $I = \int_0^{\pi/4} e^{-6x}\cos(2x)\,dx$ using Gaussian quadrature with $n = 2$, the nodes $t_1 = -0.577$, $t_2 = 0.577$ and coefficients $c_1 = c_2 = 1$. All your solutions should be expressed correct to three decimal points. (5)

Question 3 (20 marks)

Let $f(x) = -x\sin(2x) + x^2, \quad x_0 = 0, \quad x_1 = 0.3, \quad x_2 = 0.7.$

(a) Find Lagrange interpolating polynomial for $f(x)$ using the three given nodes. (4)

Solution:

$f(0) = 0, f(0.3) = -0.0793927; f(0.7) = -0.199815$

$L_0(x) = \frac{(x-0.3)(x-0.7)}{(0-0.3)(0-0.7)} = 4.76190(x-0.7)(x-0.3)$

$L_1(x) = \frac{(x-0)(x-0.7)}{(0.3-0)(0.3-0.7)} = -8.33333(x-0.7)x$

$L_2(x) = \frac{(x-0)(x-0.3)}{(0.7-0)(0.7-0.3)} = 3.57143(x-0.3)x$

$P_2(x) = f(0)L_0(x) + f(0.3)L_1(x) + f(0.7)L_2(x)$

$= 0.661606(x-0.7)x - 0.713624(x-0.3)x = -0.0520181x^2 - 0.249037x$

(b) Using the nodes x_0 and x_1, construct the Hermite interpolating polynomial $H_3(x)$ for $f(x)$ using the Lagrange coefficient polynomials. (5)

Solution:

$f(0) = 0, f(0.3) = -0.0793927$

$f'(x) = 2x - \sin(2x) - 2x\cos(2x), f'(0) = 0, f'(0.3) = -0.459844$

$L_0(x) = \frac{(x-0.3)}{(0-0.3)} = -3.33333(x-0.3)$

$L_1(x) = \frac{(x-0)}{(0.3-0)} = 3.33333x$

$L_0'(x) = -3.33333, \quad L_1'(x) = 3.33333$

$H_0(x) = [1 - 2(x-0)L_0'(0)]L_0^2 = 11.1111(x-0.3)^2(6.66667x+1)$

$H_1(x) = [1 - 2(x-0.3)L_1'(0)]L_1^2 = 11.1111(1 - 6.66667(x-0.3))x^2$

$\hat{H}_0(x) = (x-0)L_0^2 = 11.1111(x-0.3)^2x$

$\hat{H}_1(x) = (x-0.3)L_1^2 = 11.1111(x-0.3)x^2$

$H_3(x) = f(0)H_0(x) + f(0.3)H_1(x) + f'(0)\hat{H}_0(x) + f'(0.3)\hat{H}_1(x)$

$= -37.037x^2(x-0.3) - 0.882142(1 - 6.66667(x-0.3))x^2 - 37.037x(x-0.3)^2$

$= -68.1931x^3 + 30.6869x^2 - 3.33333x$

(c) Determine the natural cubic spline that interpolates the data (7)

$$f(0) = 1,\ f(3) = 2,\ f(8) = 3$$

and find the approximate value of $f(0.2)$.

Solution:

$$\begin{aligned}
S_0(x) &= a_0 + b_0x + c_0x^2 + d_0x^3, \\
S_1(x) &= a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3, \\
h_0 &= 3, h_1 = 5, a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3, c_0 = c_2 = 0
\end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 16 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ \frac{3(a_2-a_1)}{h_1} - \frac{3(a_1-a_0)}{h_0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{5} \\ 0 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}$$

So from $\mathbf{Ax} = \mathbf{b}$, we have $16c_1 = -\frac{2}{5}$ implying $c_1 = -0.025$; and the rest of the parameters are given in the table below

a_0	b_0	c_0	d_0	a_1	b_1	c_1	d_1
1	0.358333	0	-0.00277778	2	0.283333	-0.025	0.00166667

$$\begin{aligned} S_0(x) &= -0.00277778x^3 + 0.358333x + 1 \text{ } on \text{ } [0,3], \\ S_1(x) &= 0.00166667(x-3)^3 - 0.025(x-3)^2 + 0.283333(x-3) + 2, \text{ } on \text{ } [3,8] \end{aligned}$$

$$S(x) = \begin{cases} S_0(x), on \ [0,1] \\ S_1(x), on \ [1,3] \end{cases} \tag{1}$$

$f(0.2) \approx S(0.2) = S_0(0.2) = 1.07164.$

(d) The cubic Legendre polynomial is $P_3(x) = \dfrac{1}{2}(5x^3 - 3x)$. Prove that it is orthogonal (over $[-1, 1]$) to all polynomials of degree 2. (4)

Solution:

Let the general polynomial of degree 2 be given by $L_2(x) = ax^2 + bx + c$,

Then

$\int_{-1}^{1} P_2(x)L_2(x)dx = \int_{-1}^{1} \dfrac{1}{2}(5x^3 - 3x)(ax^2 + bx + c)dx$

$= \displaystyle\int_{-1}^{1} \left(\frac{5ax^5}{2} - \frac{3ax^3}{2} + \frac{5bx^4}{2} - \frac{3bx^2}{2} + \frac{5cx^3}{2} - \frac{3cx}{2} \right) dx$

$= \left[\dfrac{1}{2} \left(\dfrac{1}{4}x^4(5c - 3a) + \dfrac{5ax^6}{6} + bx^5 - bx^3 - \dfrac{3cx^2}{2} \right) \right]_{-1}^{1}$

$= \left[\dfrac{1}{2} \left(\dfrac{1}{4}(5c - 3a) + \dfrac{5a}{6} + b - b - \dfrac{3c}{2} \right) \right] - \left[\dfrac{1}{2} \left(\dfrac{1}{4}(5c - 3a) + \dfrac{5a}{6} - b + b - \dfrac{3c}{2} \right) \right]$

$= 0$

Therefore $P_2(x)$ is orthogonal to all polynomials of order 2.

Question 4 (20 marks)

(a) Develop a first-order method for approximating $f''(x)$ which uses the data $f(x - 3h)$, $f(x + 2h)$, $f(x)$ and $f(x + 5h)$. (13)

(b) Using the most accurate centered difference formula, approximate $f''(1)$ and then state the error in the approximation, given $f(x) = \sec(x) + x$, for each $h \in \{0.1, 0.01, 0.001\}$. Use an accuracy of 6 digits throughout. (7)

Question 5 (20 marks)

(a) Given the initial value problem (10)

$$x + y + \sin(x + y) + \cos(xy) - \frac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad y(0) = 0,$$

approximate $y(1)$, with $h = 0.1$ using Euler's Method.

(b) The Runge-Kutta method of order 2 (RK2) with $h = 0.1$ is used to solve (10)

$$\frac{dy}{dx} = xy$$

with $y(0) = 1$ in order to find $y(0.3)$ correct to four decimal places. Assuming that the local error in RK2 is given by

$$\epsilon_{i+1} = \frac{h^3}{6} y'''(\xi),\ \xi \in [x_i, x_{I+1}],$$

estimate an upper bound for the global error at $x = 0.3$.

Solution:

$\Delta_3 = \epsilon_3 + \alpha_2\epsilon_2 + \alpha_2\alpha_1\epsilon_1$ so that

$|\Delta_3| \leq \max\limits_{[0,0.3]} |\epsilon_m|(1 + \alpha + \alpha^2) = \max\limits_{[0,0.3]} |\epsilon_m| \left(\dfrac{\alpha^3 - 1}{\alpha - 1}\right)$ where

$\alpha = \max\limits_{[0,0.3]} |\alpha_m| = 1 + h \max\limits_{[0,0.3]} |F_y|, \quad h = 0.1.$

$\max\limits_{[0,0.3]} |\epsilon_m| = \max\limits_{[0,0.3]} \left|\dfrac{h^3}{6} y'''\right| = \max\limits_{[0,0.3]} \left|\dfrac{h^3}{6}(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2)\right|$

$f_x = y, f_y = x, f_{xx} = f_{yy} = 0, \quad f_{xy} = 1$

$$f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 = x^3 y + 3xy$$

$F(x,y) = \dfrac{1}{2} f(x,y) + \dfrac{1}{2} f(x+h, y + hf(x,y))$

$= \frac{1}{2}h^2 xy + \frac{1}{2}hx^2 y - \frac{hxy}{2} + \frac{hy}{2} + xy - \frac{y}{2}$

$F_y = \frac{h^2 x}{2} + \frac{hx^2}{2} - \frac{hx}{2} + \frac{h}{2} + x - \frac{1}{2}$

From RK2

i	0	1	2	3
x_i	0	0.1	0.2	0.3
$f(x_i)$	1	0.955	0.921193	0.897518

$\alpha = 1 + h \max\limits_{[0,0.3]} |F_y| = 1 + h \max\{0.45, 0.354, 0.257, 0.159\} = 1.045$

$\max\limits_{[0,0.3]} |\epsilon_m| = \max\limits_{[0,0.3]} \left|\dfrac{h^3}{6}(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2)\right|$

$= \dfrac{h^3}{6} \max\{0, 0.287455, 0.560085, 0.832\} = \dfrac{h^3}{6}(0.832) = 0.000138667$

$|\Delta_3| \leq \max\limits_{[0,0.3]} |\epsilon_m| \left(\dfrac{\alpha^3 - 1}{\alpha - 1}\right) = (0.000138667)\left(\dfrac{1.045^3 - 1}{1.045 - 1}\right) = 0.000435001$

END OF QUESTION PAPER