APPLIED MATHEMATICS

Introduction to Differential Equations
APM02A2/APM2A10 Main Examination
03/06/2021

Duration: 180 minutes + 30 minutes for scanning and upload **Marks:** 100
Assessor: DR F Chirove and Dr MV Visaya
Moderator: Prof E Momoniat

Instructions:

1. If you answer a question paper that you did not download yourself then you get ZERO mark automatically.
2. Carefully read and follow the instructions of each question.
3. Answer the questions in order (from 1 to 5) and write out your solutions on sheets of paper. Cross out answers that are not to be marked. Your assigned number, surname and student number must be written at the top of each page.
4. All calculations must be shown.
5. Use a scanning app (CamScanner is a good option) to scan your solutions into a PDF. Your solutions must be one PDF. Pages must be oriented correctly, i.e. not upside down or on their sides. Do not upload JPEG files.
6. Check that your PDF is not too big (it should be around 1MB/page). SAVE YOUR PDF AS "assignednumber-surname" eg 123-Visaya
7. To submit - Scroll down to "ASSIGNMENT SUBMISSION" and then "Attach files". Select your PDF and click "Submit".

Question 1 (20 marks)

(a) Consider the differential equation (DE)

$$f(x,y) = \sqrt{x+2y}$$
$$f(x,y) = \sqrt{x+3y}$$
$$f(x,y) = \sqrt{x+4y}$$

(i) Without solving the DE, plot direction fields on the x-axis, the y-axis, and $y = x$. Sketch a graph of any solution curve below $y = x$.

(ii) For what choices of (x_0, y_0) will the the initial value problem

$$f(x,y) = \sqrt{x+2y}, \quad y(x_0) = y_0.$$
$$f(x,y) = \sqrt{x+3y}, \quad y(x_0) = y_0$$

$$f(x,y) = \sqrt{x+4y}, \quad y(x_0) = y_0$$

have a unique solution on an open interval (a,b) that contains x_0?

(b) Given the differential equation (DE)

$$e^x(x^2y^2 + 2xy^2 + 6x)dx + (2x^2ye^x + 2)dy = 0,$$

$$(2x - 2y^2)dx + (12y^2 - 4xy)dy = 0,$$

$$(y^3 - 1)e^x dx + 3y^2(e^x + 1)dy = 0,$$

show that the DE is exact. Use the method for exact equations to solve the DE.

Question 2 (20 marks)

Consider the initial value problem (IVP) given by

$$y'' - 4y = e^{2x},\ y(0) = 1,\ y'(0) = -4. \quad (*)$$

$$y'' - 10y' + 25y = e^{5x},\ y(0) = -1,\ y'(0) = 1. \quad (*)$$

$$y'' - y' = 1,\ y(0) = 10,\ y'(0) = 1. \quad (*)$$

(a) Give the two IVPs of the associated homogeonous DE ($y_h(x)$) and nonhomogeneous DE ($y_p h(x)$) needed to solve $(*)$.

(b) Using your answers in (a),

(i) solve for $y_h(x)$.

(ii) solve $y_p h(x)$ using Green's function.

Question 3 (20 marks)

Consider the Cauchy-Euler equation

$$x^2y'' - 3xy' + 4y = x + x^2\ln(x), \quad x > 0.$$

$$x^2y'' - 5xy' + 9y = x + x^3\ln(x), \quad x > 0.$$

$$x^2y'' - 9xy' + 25y = x + x^5\ln(x), \quad x > 0.$$

Solve the equation by first using an appropriate substitution to convert it to a DE with constant co-efficients and using the method of variation of parameters to solve the resultant DE.

Question 4 (20 marks)

(a) In a certain culture of bacteria the rate of increase is proportional to the number present.

(i) If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?

(ii) If there are 10^4 at the end of 3 hours and 4×10^4 at the end of 5 hours, how many were there in the beginning?

(b) According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 30° and the substance cools from 100° to 70° in 15 minutes, find when the temperature will be 40°.

(c) A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

Question 5 (20 marks)

(a) Find the solution of the differential equation using Laplace transform

$$ty'' + y' + 4ty = 0,\ y(0) = 3,\ y'(0) = 0.$$

$$-ty'' - y' + 4ty = 0,\ y(0) = 4,\ y'(0) = 0.$$

$$-ty'' - y' - 4ty = 0,\ y(0) = 5,\ y'(0) = 0.$$

(b) Given

$$f(t) = \sin(-4t)\mathcal{U}(t - \frac{\pi}{4}) \text{ and } g(t) = \sin(-4t) \star \mathcal{U}(t - \frac{\pi}{4})$$

$$f(t) = \cos(-4t)\mathcal{U}(t - \frac{\pi}{3}) \text{and } g(t) = \cos(-4t) \star \mathcal{U}(t - \frac{\pi}{3})$$

$$f(t) = \cosh(-4t)\mathcal{U}(t - \frac{\pi}{6}) \text{and } g(t) = \cosh(-4t) \star \mathcal{U}(t - \frac{\pi}{6})$$

where $\mathcal{U}(t - a)$ is the unit step function.

(i) Use a translation theorem to find the Laplace transform of f.

(ii) Use the Convolution Theorem to find the Laplace transform of g.

(c) Use Laplace transforms to solve the system of differential equations.

$$\begin{aligned} x'' + y'' &= e^{2t}, \\ 2x' + y'' &= -e^{2t}, \\ x(0) = x'(0) = y(0) &= y'(0) = 0. \end{aligned}$$

$$\begin{aligned} x'' + y'' &= e^{3t}, \\ 3x' + y'' &= -e^{3t}, \\ x(0) = x'(0) = y(0) &= y'(0) = 0. \end{aligned}$$

$$\begin{aligned} x'' + y'' &= e^{5t}, \\ 5x' + y'' &= -e^{5t}, \\ x(0) = x'(0) = y(0) &= y'(0) = 0. \end{aligned}$$

END OF QUESTION PAPER