UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

PHYSICS	**AUCKLAND PARK KINGSWAY CAMPUS**

PHY8X11

EXAMINATION
23 SEPTEMBER 2021
9:30-12:00

PHY8X11

EXAMINER:	**Prof H Winkler**
EXTERNAL EXAMINER:	Prof M Chithambo Rhodes University
TIME: 2 ½ HOURS	**MARKS: 100**

Please read the following instructions carefully:

ANSWER ALL QUESTIONS: 1-3

Constants:

$c = 3 \times 10^{8}$ m.s^{-1} $\quad$ $e = 1.6 \times 10^{-19}$ C $\quad$ $h = 6.626 \times 10^{-34}$ J.s

$\hbar = h/2\pi$ $\quad$ $m_e = 9.31 \times 10^{-31}$ kg $\quad$ $m_p = 1.67 \times 10^{-27}$ kg

$a = 5.29 \times 10^{-11}$ m $\quad$ 1 eV $= 1.6 \times 10^{-19}$ J $\quad$ $\varepsilon_0 = 8.85 \times 10^{-12}$ C^{2}m^{-2}N^{-1}

$k = 1.38 \times 10^{-23}$ J.K^{-1}

Formulae:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad p_x = -i\hbar\frac{\partial}{\partial x} \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1$$

$$E_g \le \left\langle \Psi \middle| \hat{H} \middle| \Psi \right\rangle \qquad V_{ee} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \qquad \Psi(x,t) = \psi(x)\exp\left(-\frac{iEt}{\hbar}\right)$$

For the ground state of a hydrogen-like atom: $\psi_{100} = A\exp\left(\frac{Zr}{a}\right)$, $E_0 = -\frac{m}{2\hbar^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2$

$$p(x) = \sqrt{2m(E - V(x))} \qquad \int_{all\ space}|\psi|^2 r^2 \sin\theta dr d\theta d\phi = 1 \qquad [\hat{a},\hat{b}] = \hat{a}\hat{b} - \hat{b}\hat{a}$$

$$\psi(x) \cong \frac{C}{\sqrt{p(x)}}\exp\left(\pm\frac{i}{\hbar}\int p(x)dx\right) \qquad T \cong e^{-2\gamma} \qquad \gamma = \frac{1}{\hbar}\int_0^a |p(x)|dx \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\frac{dc_a}{dt} = -\frac{i}{\hbar}\left[c_b H'_{ab}\exp(-i\omega_0 t)\right] \qquad \frac{dc_b}{dt} = -\frac{i}{\hbar}\left[c_a H'_{ba}\exp(i\omega_0 t)\right] \qquad \Delta E = hc/\lambda$$

$$\hat{H} = \hat{H}_0 + \hat{H}'(t) \qquad H'_{ij} = \left\langle \psi_i \middle| H' \middle| \psi_j \right\rangle \qquad P_{a\to b}(t) = |c(t)|^2 = \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2\left(\frac{1}{2}(\omega_0 - \omega)t\right)}{(\omega_0 - \omega)^2}$$

$$\wp = e\left\langle \psi_b \middle| z \middle| \psi_a \right\rangle \qquad u = \frac{\varepsilon_0}{2}E_0^2 \qquad P_{b\to a}(t) = \frac{\pi|\wp|^2}{\varepsilon_0\hbar^2}\rho(\omega_0)t \qquad A = \frac{\omega^3\hbar}{\pi^2 c^3}B_{ba}$$

$$\hat{L}_z f = \hbar m f \qquad L^2 f = \hbar^2 l(l+1) f \qquad \sigma = \frac{1}{k^2}\sum_{l=0}^{\infty}|C_l|^2 \qquad \frac{n_1}{n_2} = \frac{g_1\exp(-E_1/kT)}{g_2\exp(-E_2/kT)}$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2} \quad n = 1,2,3,\ldots \qquad \rho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1}$$

$$f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0 \qquad D(\theta) = |f(\theta,\phi)|^2 = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\psi(r,\theta) = \psi_{incident} + \psi_{scattered} \cong A\left[\exp(ikz) + f(\theta)\frac{\exp(ikr)}{r}\right]$$

Potentially useful mathematical identities:

$\sin(a+b) = \sin a \cos b + \sin b \cos a$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$a^2 = b^2 + c^2 - 2bc\cos\theta$ sin (2a) = 2 sin a cos a

$\sin\theta = \dfrac{\exp(i\theta) - \exp(-i\theta)}{2i}$ $\cos\theta = \dfrac{\exp(i\theta) + \exp(-i\theta)}{2}$

$\int_{-\infty}^{\infty} \exp\left(-ax^2\right)dx = \sqrt{\dfrac{\pi}{a}}$ $\int_{-\infty}^{\infty} x^2 \exp\left(-ax^2\right)dx = \dfrac{1}{2a}\sqrt{\dfrac{\pi}{a}}$ $\int_0^{\pi} \sin^2 x \, dx = \dfrac{\pi}{2}$

$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

QUESTION 1 **[34]**

a) Estimate the upper limit of the ground state energy of the infinite square well potential ($V(x) = 0$ between $x = 0$ and $x = a$, and infinite elsewhere) using the variational method with the test function

$$\Psi = Ax(a-x).$$

How does this value compare with the actual ground state energy $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$? (10)

b) Briefly explain why $\Psi = A\exp\left(-2\dfrac{r_1+r_2}{a}\right)$ is a sensible test function to determine the ground state energy of a helium atom using the variational method. (5)

c) What are the assumptions underpinning the WKB approximation? (3)

d) In the event that the particle is bound in a potential that is not infinite outside the range $x = a$ to $x = b$ where $E > V(x)$, the following relationship applies:

$$\int_a^b p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar \qquad \text{(proof not needed).}$$

Use this to estimate the energies of a harmonic oscillator, i.e. a particle trapped inside a potential $V(x) = \dfrac{1}{2}m\omega^2x^2$. (9)

e) Draw the potential function for alpha particles in the vicinity of an atomic nucleus and hence briefly explain why under some circumstances a nucleus may eject an alpha particle in what is referred to as alpha decay. (7)

QUESTION 2 **[33]**

a) Briefly discuss the nature of the wave function of a two-level system that is subject to a Hamiltonian with a time-dependent perturbation term. (4)

b) Consider an electron in the ground state of a hydrogen atom, subjected to an electric field potential of the form

$$V(t) = -eE_0 z\exp\left(-t/\tau\right).$$

i) Why would transitions between the ground state to the 2p state (nlm = 210) be far more common than transitions between other states? (4)

ii) Hence explain (without doing any actual calculations) how you would use the 1-st order 2-level system approximation to determine the probability (as a function of time) that the electron undergoes a transition to the 2p state. (6)

c) In a two-level system, the equation relating the populations N of ground state "a" and excited state "b" is

$$\frac{dN_b}{dt} = -N_bA - N_bB_{ba}\rho(\omega_0) + N_aB_{ab}\rho(\omega_0)$$

where ρ is the radiation field density, and A, B_{ba} and B_{ab} are the Einstein coefficients.

/…Q2c (continued)

Through comparison with the Planck and Boltzmann laws (both listed under the formulas), confirm that

$$B_{ab} = B_{ba} \qquad \text{and} \qquad A = \frac{\omega^3 \hbar}{\pi^2 c^3} B_{ba} \tag{9}$$

d) Consider a hydrogen-like atom where the wave function takes the form ψ_{nlm}.

i) Given that $[L_z, z] = 0$, show that $\langle \psi_{n'l'm'} | z | \psi_{nlm} \rangle = 0$ unless $m' = m$. (6)

ii) State another selection rule linked to the quantum number l, and explain the implication of such selection rules for transitions between different states of this atom. (4)

QUESTION 3 [33]

a) Consider a very small particle scattering elastically off a hard sphere of radius R.

i) Show that the scattering angle equals

$$\theta = \begin{cases} 2\cos^{-1}(b/R) & b < R \\ 0 & b > R \end{cases}. \tag{6}$$

ii) Hence determine the differential cross section for this scenario. (4)

iii) Finally, confirm that the cross section is πR^2. (3)

b) In quantum mechanical scattering theory, the scattered particle's wave function is approximated by:

$$\psi(r,\theta) \cong A\left[\exp(ikz) + f(\theta)\frac{\exp(ikr)}{r}\right]$$

Briefly explain and justify the terms inside the square bracket. (6)

c) The integral form of the Schrödinger equation stipulates that the wave function can be written as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0 \quad .$$

where ψ_0 is the solution to the free particle. Show that for the common case that $|\mathbf{r}| >> |\mathbf{r}_0|$, the scattering amplitude therefore may be approximated by

$$f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0 \tag{9}$$

d) Consider the expression for the scattering amplitude given by

$$f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0 \ .$$

Without doing any calculations, explain in each case under what condition the following approximations are valid:

i) $f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int V(\mathbf{r})d^3\mathbf{r}$, (3)

ii) $f(\theta) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(\kappa r)dr$. (2)

END