PHY8X11 – 2021 – Exam – Answers and marking (NOT a model solution for rote learning)

QUESTION 1 [34]

a) Estimate the upper limit of the ground state energy of the infinite square well potential ($V(x) = 0$ between $x = 0$ and $x = a$, and infinite elsewhere) using the variational method with the test function

$$\Psi = Ax(a-x) \, . \qquad \text{[class example]}$$

How does this value compare with the actual ground state energy $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$? (10)

$\langle\Psi\vert\Psi\rangle = A^2\int_0^a x^2(a-x)^2\,dx = ...,$ … (after some calculations) … energy $= \dfrac{5\hbar^2}{ma^2}$

b) Briefly explain why $\Psi = A\exp\left(-2\dfrac{r_1+r_2}{a}\right)$ is a sensible test function to determine the ground state energy of a helium atom using the variational method. [conceptual] (5)

Give a detailed, physical, sensible and understandable explanation of this

c) What are the assumptions underpinning the WKB approximation? [conceptual] (3)

Give a detailed, physical, sensible and understandable explanation of this

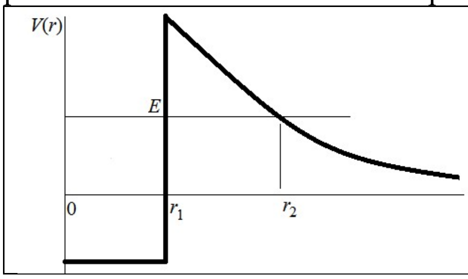
d) In the event that the particle is bound in a potential that is not infinite outside the range $x = a$ to $x = b$ where $E > V(x)$, the following relationship applies:

$$\int_a^b p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar \qquad \text{(proof not needed).}$$

Use this to estimate the energies of a harmonic oscillator, i.e. a particle trapped inside a potential $V(x) = \dfrac{1}{2}m\omega^2x^2$. [assignment] (9)

$p(x) = \sqrt{2m(E-V(x))} = ...,$ … (after some calculations) … $\Rightarrow E = \left(n - \dfrac{1}{2}\right)\hbar\omega$

e) Draw the potential function for alpha particles in the vicinity of an atomic nucleus and hence briefly explain why under some circumstances a nucleus may eject an alpha particle in what is referred to as alpha decay. [conceptual] (7)



Give a detailed, physical, sensible and understandable explanation of what is going on

QUESTION 2 [33]

a) Briefly discuss the nature of the wave function of a two-level system that is subject to a Hamiltonian with a time-dependent perturbation term. [conceptual] (4)

Give a detailed, physical, sensible and understandable explanation of this

b) Consider an electron in the ground state of a hydrogen atom, subjected to an electric field potential of the form $V(t) = -eE_0 z \times \exp(-t/\tau)$.

i) Why would transitions between the ground state to the 2p state (nlm = 210) be far more common than transitions between other states? (4)

ii) Hence explain (without doing any actual calculations) how you would use the 1-st order 2-level system approximation to determine the probability (as a function of time) that the electron undergoes a transition to the 2p state. [assignment] (6)

i) Give a detailed, physical, sensible and understandable explanation of this
ii) Substitute the wave functions , … (some calculations) … Finally … $P(t) = \lvert c_b(t)\rvert^2$.

c) In a two-level system, the equation relating the populations N of ground state "a" and excited state "b" is

$$\frac{dN_b}{dt} = -N_b A - N_b B_{ba}\rho(\omega_0) + N_a B_{ab}\rho(\omega_0)$$

where ρ is the radiation field density, and A, B_{ba} and B_{ab} are the Einstein coefficients.

Through comparison with the Planck and Boltzmann laws (both listed under the formulas), confirm that

$$B_{ab} = B_{ba} \quad \text{and} \quad A = \frac{\omega^3 \hbar}{\pi^2 c^3} B_{ba} \qquad \text{[derivation]} \quad (9)$$

In equilibrium, $\frac{dN_b}{dt} = \ldots = 0$, (after some calculations) … $B_{ab} = B_{ba}$ and $A = \frac{\omega^3 \hbar}{\pi^2 c^3} B_{ba}$

d) Consider a hydrogen-like atom where the wave function takes the form ψ_{nlm}.

i) Given that $[L_z, z] = 0$, show that $\langle \psi_{n'l'm'} | z | \psi_{nlm} \rangle = 0$ unless $m' = m$. [derivation] (6)

ii) State another selection rule linked to the quantum number l, and explain the implication of such selection rules for transitions between different states of this atom. (4)

i) $0 = \langle \psi_{n'l'm'} \vert [L_z, z] \vert \psi_{nlm} \rangle = \ldots$, … (after some calculations) …
$\Rightarrow \langle \psi_{n'l'm'} \vert z \vert \psi_{nlm} \rangle = 0$ or $m' = m$
ii) Give a detailed, physical, sensible and understandable explanation of this

QUESTION 3 [33]

a) Consider a very small particle scattering elastically off a hard sphere of radius R.

i) Show that the scattering angle equals

$$\theta = \begin{cases} 2\cos^{-1}(b/R) & b < R \\ 0 & b > R \end{cases} \qquad (6)$$

ii) Hence determine the differential cross section for this scenario. (4)

iii) Finally, confirm that the cross section is πR^2. [derivation, class example] (3)

i) If $b > R$ then , … (after some calculations) … $\therefore \theta = \begin{cases} 2\cos^{-1}(b/R) & b < R \\ 0 & b > R \end{cases}$ ii) $\theta = \ldots \Rightarrow D(\theta) = \ldots = \dfrac{R^2}{4}$ iii) $D(\theta) = \dfrac{d\sigma}{d\Omega} \Rightarrow \sigma = \ldots = \pi R^2$

b) In quantum mechanical scattering theory, the scattered particle's wave function is approximated by:

$$\psi(r,\theta) \cong A\left[\exp(ikz) + f(\theta)\frac{\exp(ikr)}{r}\right]$$

Briefly explain and justify the terms inside the square bracket. (6)

Give a detailed, physical, sensible and understandable explanation of this

c) The integral form of the Schrödinger equation stipulates that the wave function can be written as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0 \ .$$

where ψ_0 is the solution to the free particle. Show that for the common case that $|\mathbf{r}| >> |\mathbf{r}_0|$, the scattering amplitude therefore may be approximated by

$$f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0$$

[derivation] (9)

When $|\mathbf{r}| >> |\mathbf{r}_0|$, then $k|\mathbf{r}-\mathbf{r}_0| \approx kr - \mathbf{k}\cdot\mathbf{r}_0$, and , … (after some calculations) … $f(\theta,\phi) \cong -\dfrac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0$

d) Consider the expression for the scattering amplitude given by

$$f(\theta,\phi) \cong -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}\cdot\mathbf{r}_0)V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0 \ .$$

Without doing any calculations, explain in each case under what condition the following approximations are valid:

i) $f(\theta,\phi) \cong -\dfrac{m}{2\pi\hbar^2}\int V(\mathbf{r})d^3\mathbf{r}$, (3)

ii) $f(\theta) = -\dfrac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(\kappa r)dr$. [conceptual] (2)

Give a detailed, physical, sensible and understandable explanation of this.