- **Question 1**(18 points)
Assume that for every T the price of a $T-$bond has the form

$$p(t,T) = F^T(t, r(t)), \tag{1}$$

and the short rate has the following SDE

$$dr(t) = \tilde{\mu}(t, r(t))dt + \sigma(t, r(t))d\tilde{W}(t), \tag{2}$$

under the observed probability measure P, and

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{3}$$

under the martingale probability measure Q

(i) Show that F^T satisfies the following PDE

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\}F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T,r) = 1 \end{cases} \tag{4}$$

(ii) Under which condition on $\mu(t, r(t))$, we have

$$F(t,r;T) = E_{t,r}^{Q}\left[e^{-\int_t^T r(s)ds}\right], \tag{5}$$

(iii) Assume that the model admits an affine term structure solution

$$F(t,r;T) = e^{A(t,T)-B(t,T)r(t)}.$$

Give sufficient conditions that A and B should satisfy to allow the existence of the affine term structure.

- **Question 2**(7 points)
Assume that following forward rate dynamic

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{6}$$

under the martingale measure.

(i) Show that the zero-coupon price has the following dynamic

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt \\ &+ p(t,T)S(t,T)dW(t) \end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds \\ S(t,T) = -\int_t^T \sigma(t,s)ds \end{cases}$$

(ii) Derive the HJM drift condition that $\alpha(t,T)$ must satisfy to have an arbitrage free model.

- **Question 3**(25 points)
  Consider the martingale measure Q and the following $Q-$dynamic for the short rate

$$dr(s) = 0.5(0.05 - r(s))ds + 0.4dW(s)$$
$$r(0) = 0.1$$

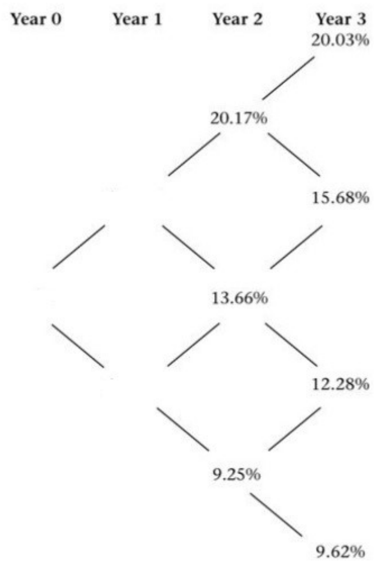
  (i) Derive the expression for the zero-coupon price $P(0,T)$.

  (ii) Give the distribution of $P(0,T)$ and its parameters

  (iii) Determine the price at time 0 of European call option with strike price 0.7 and exercise date 1, on an underlying 2-year bond.

  (iv) Determine the price at time 0 of a two-year floor with strike rate 0.05 and a notional value 10,000. The payments are made annually.

- **Question 4**(25 points)
  You are given the following incomplete Black-Derman-Toy interest rate tree model for the effective annual interest rates



We also have the following market data

Maturity (years)	Yield to Maturity	Bond Price ($)	Volatility in Year 1
1	10%	0.90901	NA
2	11%	0.8116	10%
3	12%	0.7118	15%
4	12.5%	0.6243	14%

  (i) Find the missing interest rates.

  (ii) Price a 2-year European call option on a 2-year zero-coupon bond with strike price $0.8.

  (iii) Price a 1-year European put option on a 3-year zero-coupon bond with strike price $0.9.

- **Question 5**(25 points)
  Consider the HJM framework under the martingale probability measure Q with the following dynamic

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + 0.2e^{-(T-t)}dW(t) \\ f(0,T) &= f^{\star}(0,T), \end{aligned}$$

  where W is a standard Brownian motion under Q.

  (i) Determine the expression of the drift for the rates $f(t,T)$ if the SDE satisfies the HJM drift condition.

  (ii) Find the distribution of $P(0,T)$ under Q.

  (iii) Price a 1-year European Put option on a 2-year zero-coupon bond with strike price \$0.75.

  (iv) Determine the price at time 0 of a two-year cap with strike rate 0.08 and a notional value 100,000. The payments are made annually.

Appendix
Let $X \sim N\left(a, b^2\right)$ and $K \in \mathbb{R}$. Then we have

$$\mathbb{E}\left[\left(e^X - K\right)^+\right] = e^{a+\frac{b^2}{2}} N\left(-\frac{\log K - \left(a + b^2\right)}{b}\right) - KN\left(-\frac{\log K - a}{b}\right)$$

$$\mathbb{E}\left[\left(K - e^X\right)^+\right] = KN\left(\frac{\log K - a}{b}\right) - e^{a+\frac{b^2}{2}} N\left(\frac{\log K - \left(a + b^2\right)}{b}\right)$$

Note that $\mathbb{E}[e^X] = e^{a+\frac{b^2}{2}}$.