PHY8X17 2020 (v1.2): Formulas and Identities

Chapter 1: Series

- Taylor series: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)$
- Shifted Taylor series: $f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$
- Maclaurin series: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$
- Power series: $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n$
- Binomial expansion: $(1+x)^m = 1 + xm + \frac{x^2}{2!}m(m-1) + \ldots + \frac{x^n}{n!}m(m-1)\ldots(m-n+1) + \ldots$
  If $m \in \mathbb{N}$: $(1+x)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}x^n = \sum_{n=0}^{\infty} \binom{m}{n} x^n$

Chapter 2: Ordinary Differential Equations

- First order ODE: $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = -\frac{P(x,y)}{Q(x,y)}$
- Separable ODE: $P(x)\mathrm{d}x + Q(y)\mathrm{d}y = 0 \Rightarrow \int_{x_0}^{x} P(x)\mathrm{d}x + \int_{y_0}^{y} Q(y)\mathrm{d}y = 0$
- Exact ODE: $\mathrm{d}\varphi = \frac{\partial \varphi}{\partial x}\mathrm{d}x + \frac{\partial \varphi}{\partial y}\mathrm{d}y = 0 \Rightarrow \varphi(x,y) = \int_{x_0}^{x} P(x,y)\mathrm{d}x + \int_{y_0}^{y} Q(x_0,y)dy = \text{constant}$
- Linear ODE: $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x) \Rightarrow y(x) = \frac{1}{\alpha(x)}\left[\int^{x} \alpha(x)q(x)\mathrm{d}x + C\right] \equiv y_2(x) + y_1(x)$
  with $\alpha(x) = \exp\left[\int^{x} p(x)\mathrm{d}x\right]$
- Second order linear ODE: $y'' + q(x)y' + q(x)y = f(x)$
- Series solution: $y\,(x) = x^s\left(a_0 + a_1 x + a_2 x^2 + \ldots\right) = \sum_{j=0}^{\infty} a_j x^{s+j}$ with $a_0 \neq 0$

Chapter 4: Green's Functions

- Green's function: $\mathcal{L}G(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) = \delta(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2)$
- ODE: $y(\vec{\mathbf{r}}_1) = \int G(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2) f(\vec{\mathbf{r}}_2)\,\mathrm{d}\tau$
- Second order ODE of form $\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = f(x)$:
  $y(x) = Ay_2(x)\int_a^x y_1(t)f(t)\,\mathrm{d}t + Ay_1(x)\int_x^b y_2(t)f(t)\,\mathrm{d}t$ with $A\left[y_2'(t)y_1(t) - y_1'(t)y_2(t)\right] = \frac{1}{p(t)}$

Chapter 5: Complex Variable Calculus

- Cauchy-Riemann conditions: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- Cauchy integral formula: $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z = \begin{cases} f(z_0), & z_0 \text{ within the contour} \\ 0, & z_0 \text{ exterior to the contour} \end{cases}$
- Derivatives: $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$
- Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ with $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\,\mathrm{d}z'$
- Residue theorem: $\oint f(z)\,\mathrm{d}z = 2\pi i\,(a_{-1,z_0} + a_{-1,z_1} + a_{-1,z_2} + \ldots) = 2\pi i \times$(sum of enclosed residues)
  with $a_{-1} = \frac{1}{(m-1)!}\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left((z-z_0)^m f(z)\right)\right]_{z=z_0}$
- Cauchy principal value: $⨍ f(x)\mathrm{d}x = \lim_{\delta\to 0^+}\int^{x_0-\delta} f(x)\mathrm{d}x + \int_{x_0+\delta} f(x)\mathrm{d}x$

Chapter 6: Orthogonal Polynomials

For ODEs of form $p(x)y'' + q(x)y' + \lambda y = 0$ with $p(x) = \alpha x^2 + \beta x + \gamma$ and $q(x) = \mu x + \nu$:

- Rodrigues formula: $y_n(x) = \frac{1}{w(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [w(x)p(x)^n] = \frac{1}{w(x)} \frac{n!}{2\pi i} \oint_c \frac{w(z)p(z)^n}{(z-x)^{n+1}} \mathrm{d}z$
with $w(x) = p^{-1} \exp\left(\int^x \frac{q(x)}{p(x)} \mathrm{d}x\right)$

- Generaing functions: $g(x,t) = \sum_n c_n f_n(x) t^n = \frac{1}{w(x)} \sum_{n=0}^{\infty} c_n t^n \frac{n!}{2\pi i} \oint_c \frac{w(z)p(z)^n}{(z-x)^{n+1}} \mathrm{d}z$

Chaper 7: Fourier Series

- Fourier Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with
$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos ns \mathrm{d}s, \quad n = 0, 1, 2, \ldots,$
$b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin ns \mathrm{d}s, \quad n = 1, 2, \ldots,$
$c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$ and $c_0 = \frac{1}{2}a_0$

Chapter 8: Integral Transforms

- Fourier transform (1D): $g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \mathrm{d}t$

- Fourier sine/cosine transform (1D): $g(\omega) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(t) \sin(\omega t) \mathrm{d}t$, $g(\omega) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(t) \cos(\omega t) \mathrm{d}t$

- Inverse Fourier transform (1D): $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \mathrm{d}\omega$

- Fourier transforms (3D): $g(\vec{\mathbf{k}}) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{\mathbf{r}}) e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \mathrm{d}^3 r$, $f(\vec{\mathbf{r}}) = \frac{1}{(2\pi)^{3/2}} \int g(\vec{\mathbf{k}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \mathrm{d}^3 k$

- Properties: $FT[f(\vec{\mathbf{r}} - \vec{\mathbf{R}})] = e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{R}}} g(\vec{\mathbf{k}})$; $FT[f(\alpha\vec{\mathbf{r}})] = \frac{1}{\alpha^3} g(\alpha^{-1}\vec{\mathbf{k}})$; $FT[f(-\vec{\mathbf{r}})] = g(-\vec{\mathbf{k}})$;
$FT[f^*(-\vec{\mathbf{r}})] = g^*(\vec{\mathbf{k}})$; $FT[\nabla f(\vec{\mathbf{r}})] = -i\vec{\mathbf{k}} g(\vec{\mathbf{k}})$; $FT[\nabla^2 f(\vec{\mathbf{r}})] = -k^2 g(\vec{\mathbf{k}})$

- Laplace transform: $f(s) = \int_0^{\infty} e^{-st} F(t) \mathrm{d}t$

- Properties: $LT\left[\int_0^t F(x) \mathrm{d}x\right] = \frac{1}{s} f(s)$; $LT[F(at)] = \frac{1}{a} f(\frac{s}{a})$; $LT[e^{at} F(t)] = f(s-a)$;
$LT[F(t-b)] = e^{-bs} f(s)$; $LT[(-t)^n F(t)] = f^{(n)}(s)$; $LT\left[\frac{F(t)}{t}\right] = \int_s^{\infty} f(x) \mathrm{d}x$;
$F(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{st} f(s) \mathrm{d}s$

Chapter 9: Calculus of Variations

- Functional to be optimised: $J[y] = \int_{x_1}^{x_2} f\left(y(x), \frac{\mathrm{d}y(x)}{\mathrm{d}x}, x\right) \mathrm{d}x$

- Euler equation: $\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y_x} = 0$ or $\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0$

- Generalised Euler equation: $\frac{\partial f}{\partial y_i} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial f}{\partial y_{ij}} = 0$ with $y_{ij} \equiv \frac{\partial y_i}{\partial x_j}$

- Lagrange multipliers f under constraints g_j: $\frac{\partial f}{\partial x_i} - \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i} = 0$

Cartesian coordinates

- $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}$
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Cylindrical coordinates

- $x = \rho\cos\phi$; $y = \rho\sin\phi$; $z = z$
- $A_\rho = A_x\cos\phi + A_y\sin\phi$
  $A_\phi = -A_x\sin\phi + A_y\cos\phi$
  $A_z = A_z$
- $\nabla\cdot\vec{\mathbf{F}} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho F_\rho\right] + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left[F_\phi\right] + \frac{\partial}{\partial z}\left[F_z\right]$
- $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial r}f\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}f + \frac{\partial^2}{\partial z^2}f$

Spherical coordinates

- $x = r\cos\phi\sin\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\theta$
- $A_r = A_x\cos\phi\sin\theta + A_y\sin\phi\sin\theta + A_z\cos\theta$
  $A_\theta = A_x\cos\phi\cos\theta + A_y\sin\phi\cos\theta - A_z\sin\theta$
  $A_\phi = -A_x\sin\phi + A_y\cos\phi$
- $\nabla\cdot\vec{\mathbf{F}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2F_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta F_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[F_\phi\right]$
- $\nabla^2 f = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}f\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}f\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}f\right]$

Identities

- $(\vec{\mathbf{A}}\times\vec{\mathbf{B}})\cdot\vec{\mathbf{C}} = \vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}})$
- $\vec{\mathbf{A}}\times(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) = \vec{\mathbf{B}}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) - \vec{\mathbf{C}}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}})$
- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla\cdot(f\vec{\mathbf{A}}) = (\nabla f)\cdot\vec{\mathbf{A}} + f(\nabla\cdot\vec{\mathbf{A}})$
- $\nabla\cdot(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}\cdot(\nabla\times\vec{\mathbf{A}}) - \vec{\mathbf{A}}\cdot(\nabla\times\vec{\mathbf{B}})$
- $\nabla\times(f\vec{\mathbf{A}}) = (\nabla f)\times\vec{\mathbf{A}} + f(\nabla\times\vec{\mathbf{A}})$
- $\nabla(1/R) = -\hat{\mathbf{R}}/r^2$, with $\vec{\mathbf{R}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}'$
- $\nabla'(1/R) = \hat{\mathbf{R}}/r^2$, with $\vec{\mathbf{R}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}'$