FACULTY OF SCIENCE

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**MAT8X13** **SET THEORY AND LOGIC**
CAMPUS	**APK**
ASSESSMENT	**EXAM**

DATE 18/06/2020	**MARKS 90**
ASSESSOR	**DR C RATHILAL**
MODERATOR	**PROF D BABOOLAL (UKZN)**
DURATION 6 HOURS	**SUBMISSION DEADLINE: 14:00**

INSTRUCTIONS:

1. ANSWER ALL QUESTIONS.
2. QUESTIONS CAN BE ANSWERED IN ANY ORDER.
3. SHOW ALL CALCULATIONS AND MOTIVATE ALL ANSWERS.
4. USING CAMSCANNER CREATE A PDF SCAN OF YOUR SOLUTIONS (USE THE MAGIC FILTER).
5. ATTACH THE SCAN TO EMAIL AND SEND TO crathilal@uj.ac.za BY 14:00.
   LATE SUBMISSIONS NOT ACCEPTED.
6. GOOD LUCK.

Question 1 [3 marks]

(a) Show that if $A \subseteq B$ then $\bigcup A \subseteq \bigcup B$. (2)

(b) Show that there is no set to which every singleton (that is, every set of the form $\{x\}$) belongs. (1)

Question 2 [5 marks]

(a) Give an example of sets a and B for which $a \in B$ but $\wp(a) \notin \wp(B)$. (1)

(b) Show that $\bigcup\{\wp(X)|X \in \mathcal{A}\} \subseteq \wp(\bigcup \mathcal{A})$.
Under what conditions does equality hold? (3)

(c) Simplify $\bigcap\{\wp(\wp(\wp(\{\emptyset\}))), \wp(\wp(\{\emptyset\})), \wp(\{\emptyset\})\}$. (1)

Question 3 [9 marks]

(a) Given sets A and B, define $A \times B$. Provide a justification for the definition. That is, explain fully with the use of axioms and results. (2)

(b) Show that the following generalisation of ordered pairs to ordered triples by defining

$$\langle x, y, z\rangle^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

is unsuccessful by giving an example. (2)

(c) For a set F, show that F^{-1} is a function iff F is single rooted. (3)

(d) Let

$$F = \{ \langle \emptyset, \{\{\emptyset\}, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle \}.$$

Evaluate the following: $F \circ F$, $F^{-1}, F[\{\emptyset\}]$, and $F(\emptyset)$. (2)

Question 4 [2.5 marks]

Consider $S \subseteq \mathbb{Q} \times \omega$ defined by $S = \left\{\left\langle \frac{a}{b}, a^2 + 2b^2\right\rangle : a, b \in \mathbb{Z}, b \neq 0 \right\}$.

(a) Is S a function from $\mathbb{Q}$ to ω? Justify your answer. (2)

(b) Give one element of the set $\omega \setminus ran(S)$. Justify your answer. (0.5)

Question 5 [9.5 marks]

(a) Show that R is transitive iff $R \circ R \subseteq R$ (2)

(b) Find 2 equivalence relations on the set $\{0, 1, 2\}$. Justify your answers. (2)

(c) Assume that R is a linear ordering on a set A. Show that R^{-1} is also a linear ordering on A. (2)

(d) Find all linear orderings on $\{0, 1, 2\}$ that contain $\langle 0, 2 \rangle$. (1.5)

(e) Simplify $\varepsilon^{-1}[\{6, 7\}]$. (2)

Question 6 [6 marks]

(a) Let $\mathcal{A}$ be a non-empty set. Assume that every member of $\mathcal{A}$ is a transitive set. Show that $\bigcap \mathcal{A}$ is a transitive set. (2)

(b) Determine if the following set is transitive : $\{0, 1, \{1\}\}$. Justify your answer. (1.5)

(c) Write an expression for the set "3" using only $\emptyset$, $\{$, $\}$, and commas. (1)

(d) Let $A = \{1\}$. Calculate A^+ and $\bigcup(A^+)$. (1.5).

Question 7 [5.5 marks]

(a) For any sets A and B, show that if $A \approx B$ then $B \approx A$. (2)

(b) State two properties of the set ω. What is card ω? Give 2 examples of infinite cardinals, other than card ω. (2)

(c) Let κ, λ and μ be cardinal numbers, and consider the following statement:

$$\kappa \leq \lambda \Rightarrow \kappa + \mu \leq \lambda + \mu.$$

Give a counter example to show that the above statement cannot be strenghtened by replacing $\leq$ by $<$. (1.5)

Question 8 [6 marks]

(a) Prove that the following statement is equivalent to the axiom of choice: For any set $\mathcal{A}$ whose members are non-empty sets, there is a function with domain $\mathcal{A}$ such that $f(X) \in X$, for all $X \in \mathcal{A}$. (3)

(b) Recall Theorem 6N from your Set Theory textbook:

$$\text{For any infinite set } A, \text{ we have } \omega \preceq A.$$

State the definitions of functions g and h from the proof of Theorem 6N and show that $g[n] = h(n)$, for $n \in \omega$. (3)

Question 9 [8 marks]

(a) Using the Regularity axiom, show that there does not exist any sets a and b with $a \in b$ and $b \in a$. (1.5)

(b) Let α be an ordinal. Show that any element of α is an ordinal. (1.5)

(c) Show that $\bigcup\{\,\omega, \omega^{+}, \omega^{++}, \omega^{+++}, \ldots\}$ is an ordinal. (2.5)

(d) Let A be a non-empty set of ordinals. Show that if A has a greatest element α, then $\bigcup A = \alpha$, otherwise $\bigcup A$ is a limit ordinal. (2.5)

Question 10 [9 marks]

(a) Without constructing a full truth table, show that the statement form $(((p \to q) \wedge p) \to q)$ is a tautology. (2)

(b) Find, systematically, a statement form in disjunctive normal form that is logically equivalent to $(p \to q) \to (\sim (q \to p))$ (2)

(c) Let Γ be a set of wfs of the classical propositional calculus L, prove that if $\Gamma \vdash_L B$ then $\Gamma \cup \{\sim B\}$ is inconsistent in L (2)

(d) Using only the DT, the IL and (generalised) modus ponens, show that $\vdash_L ((\sim B) \to (\sim A)) \to (A \to B)$ (3)

In the remaining questions, T denotes a first order theory (with equality), and A, B denotes wfs of T.

Question 11 [10 marks]

(a) Using the axioms and the rules of T, and the Theorem on Tautologies (but not the completeness Theorem for T), show that

$$\vdash_T (\sim (x = z) \rightarrow (\forall y)[(x = y) \rightarrow (\sim (y = z))].$$

Hint: Let A abbreviate $\sim (x = z)$ (3)

(b) Prove the Validity Theorem: The theorems of T are valid in every model of T. (4)

(c) Let G be a first order theory for groups and let A be the wf $x = 0$. Use the Validity Theorem to show that $\nvdash_G A$. Be precise. (3)

Question 12 [9 marks]

(a) Prove that $\vdash_T ((\forall x)B) \rightarrow (\sim (\exists x)(\sim B)))$. Do no use the Completeness Theorem. (3)

(b) Determine if the following statement is logically valid: $((\forall x)(\exists y)A) \rightarrow ((\exists y)(\forall x)A)$. (3)

(c) Prove that if we drop the requirement in an (A4) axiom that A admits t for x, then we lose the Validity Theorem. (3)

Question 13 [7.5 marks]

(a) Let Γ be a set of wfs of T. Show that Γ has a model if and only if every finite subset of Γ has a model. (2.5)

(b) Recall that if T is an axiomatic first order theory, then its axioms can be enumerated effectively, i.e., there is a mechanical procedure for producing a list $A_1, A_2, A_3, \ldots$ of all axioms of T. Explain why, in this case, the theorems of T can also be enumerated effectively. (2)

(c) Let N denote Peano's first order theory of arithmetic. Assume that N is consistent. Deduce from Gödels first Incompleteness Theorem that there exists a closed wf A of N and a model $\mathcal{S}$ of N such that A is true in $\mathcal{S}$ but false in the standard model $\mathcal{N}$ of N. (3)