# University of Johannesburg

## FACULTY OF SCIENCE

**Pure and Applied Mathematics (APK)**
**MAT8X01**
**Graph Theory A**
**SUPPLEMENTARY EXAM**
**July 2020**

Examiner .................................... Prof P. Dankelmann
External Examiner ...... Dr. D. Erwin (University of Cape Town)
Time ................................................ 4 hours
Total marks ............................................ 50

Please read the following instructions carefully:

1) Answer all questions.
2) This paper consists of this cover page and two pages of questions.

# **Graph Theory Honours MAT8X01**

Supplementary Exam

Total marks: 50 .............................. July 2020

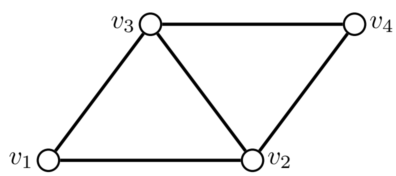
**Question 1** [3+2+2.5]

(a) Determine if the following sequences are graphical. For each graphical sequence construct a graph having this sequence as a degree sequence:

(i) 4,4,2,2,1,1, (ii) 5,5,4,3,2,1.

(b) Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence. Prove that the sequence $s' : d_1, d_1, d_2, d_2, d_3, d_3, \ldots, d_n, d_n$, that is the sequence of length $2n$ in which every entry of $s$ is repeated, is graphical.

(c) Use the adjacency matrix of the graph below to determine the number of walks of length 4 from $v_1$ to $v_2$.




**Question 2** [2+1.5+2]

(a) Let $G$ be a graph of order $n$ and size $n-1$ with the property that removing any edge increases the number of components of $G$. Prove that $G$ is a tree.

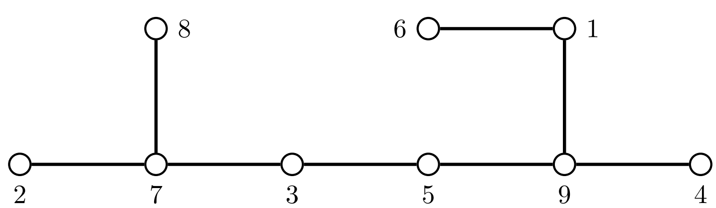
(b) Let $T$ be a tree. 75% of the vertices of $T$ have degree 1, while the remaining 25% have degree 3. Determine the order of $T$.

(c) Decide for each of the following sequences if it is the degree sequence of a tree. If it is, is every graph with this degree sequence a tree? Give reasons.

(i) $4, 4, 2, 2, 1, 1, 1, 1, 1, 1$, (ii) $4, 4, 2, 2, 1, 1$.

**Question 3** [2.5+3+2]

(a) Let $T$ be the tree below. Determine its Prüfer code.

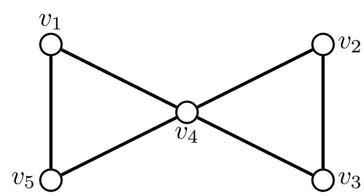



(b) Determine the labelled tree with Prüfer code 1515. Show your working.

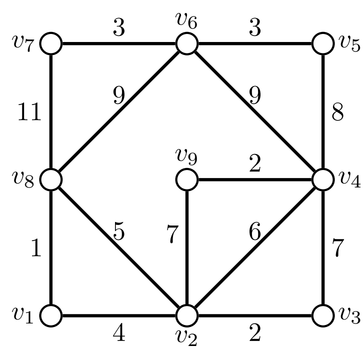
(c) Determine the number of labelled trees with vertex set $\{1, 2, \ldots, n\}$ in which vertices 1 and 2 are leaves.

**Question 4** [2.5+2.5]

(a) Use the matrix-tree theorem to determine the number of spanning trees of the graph below.




(b) Use Kruskal's algorithm to find a spanning tree of minimum weight of the following weighted graph. Show every step of the algorithm.




**Question 5** [3.5+2+2.5]

(a) Let $G$ be a $k$-connected graph, $k \geq 2$, and let $v_1, v_2, \ldots, v_k$ be $k$ different vertices. Assume that $G$ has a cycle $C$ that contains $v_1, v_2, \ldots, v_{k-1}$. Show that $G$ contains a cycle that contains $v_1, v_2, \ldots, v_k$ (and possible other vertices).

(b) Let $G$ be a graph of order 21, minimum degree 10 and maximum degree 20. Determine $\lambda(G)$. Give two different reasons for your answer.

(c) True or false? Let $k \in \mathbb{N}$ with $k \geq 2$. If $G$ is a $k$-connected graph and $u, v, w$ three distinct vertices of $G$, then there exist $k-1$ internally disjoint paths from $u$ to $v$, none of which contains $w$. Give a proof or a counter-example.

**Question 6** [2.5+2.5]

(a) Let $G_1$ and $G_2$ be two vertex disjoint connected graphs. Let $v_1$ be a vertex of $G_1$ and $v_2$ a vertex of $G_2$. Is the following statement always true, sometimes true, or always false? *The graph $H$, obtained from $G_1$ and $G_2$ by adding the edge $v_1v_2$, is Eulerian.* Give reasons for your answer.

(b) Let $G$ be a 3-regular graph. Assume that it is possible to add three edges to $G$ to obtain an Eulerian graph. Determine the number of vertices of $G$.

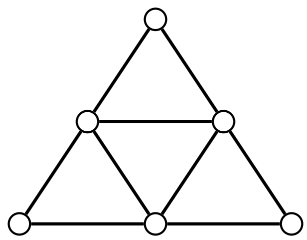
**Question 7** [1.5+1+2.5+2]

(a) Let $a \in \mathbb{N}$ with $a \geq 2$. Let $G$ be the complete bipartite graph $K_{a,a}$. Prove that the closure of $G$ is complete.

(b) Let $a \in \mathbb{N}$ with $a \geq 1$ Give two different proofs that the complete bipartite graph $K_{a,a+1}$ has a Hamilton path.

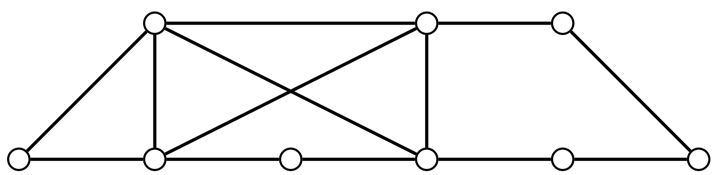
(c) Give an example of a 3-connected graph that does not have a Hamilton cycle. Prove that your graph does not have a Hamilton cycle.

(d) Determine the toughness of the following Hamiltonian graph. Give reasons.



**Question 8** [2+2+2]

(a) Decide if the following graph has a strong orientation. If yes, find such an orientation.



(b) True or false? Every Eulerian graph has a strong orientation. Give reasons.

(c) Decide if the graph in (a) has an Eulerian orientation. If no, give reasons. If yes, find such an orientation.