University of Johannesburg

FACULTY OF SCIENCE

Pure and Applied Mathematics (APK)
MAT8X01
Graph Theory A
EXAM
June 2020

Examiner .... Prof P. Dankelmann
External Examiner ...... Dr. D. Erwin (University of Cape Town)
Time .... 4 hours
Total marks 50

Please read the following instructions carefully:

1) Answer all questions.
2) This paper consists of this cover page and two pages of questions.

Graph Theory Honours MAT8X01

Total marks: 50 .... June 2020

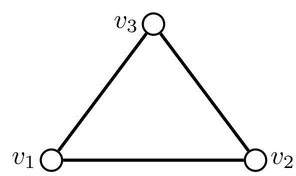
Question 1 [3+2+2.5]

(a) Determine if the following sequences are graphical. For each graphical sequence construct a graph having this sequence as a degree sequence:

(i) 4,4,3,3,3,1, (ii) 5,5,4,4,3,1.

(b) Let $s : d_1, d_2, \ldots, d_n$ be a graphical sequence. Prove that the sequence $s' : n-1-d_n, n-1-d_{n-1}, \ldots, n-1-d_1$ is graphical.

(c) Use the adjacency matrix of the graph below to determine the number of walks of length 4 from v_1 to v_2.



Question 2 [2+1+2]

(a) Let T_1 and T_2 be two vertex disjoint trees. Let T be the graph obtained from T_1 and T_2 by choosing a vertex v_1 of T_1 and a vertex v_2 of T_2, then adding a new edge v_1v_2. In other words, $V(T) = V(T_1) \cup V(T_2)$ and $E(T) = E(T_1) \cup E(T_2) \cup \{v_1v_2\}$. Prove that T is a tree.

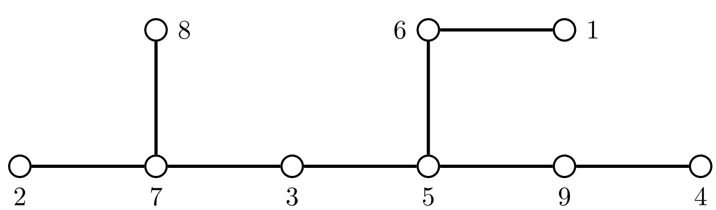
(b) Let T be a tree of order 9. The degrees of eight vertices are given: One vertex has degree 5, three vertices have degree 2, four vertices have degree 1. What is the degree of the ninth vertex? Give reasons.

(c) Decide for each of the following sequences if is the degree sequence of a tree. If it is, is every graph with this degree sequence a tree? Give reasons.

(i) $4, 3, 2, 1, 1, 1, 1, 1$, (ii) $3, 3, 3, 2, 1, 1, 1$.

Question 3 [2.5+3+2.5]

(a) Let T be the tree below. Determine its Prüfer code.

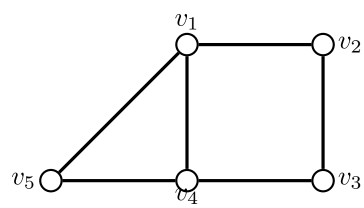


(b) Determine the tree with Prüfer code $1, 5, 5, 5$.

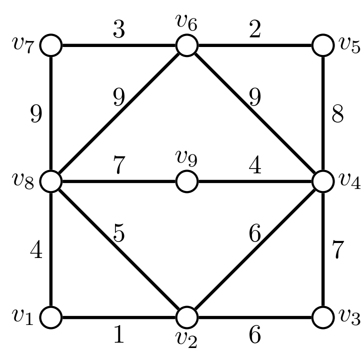
(c) Let T be a labelled tree. Assume that the Prüfer code of T is known, except for one missing entry: $1, 3, 5, 5, \_\_, 4, 9, 6$. List all possible values for the missing entry? Give reasons for your answer.

Question 4 [2.5+2.5]

(a) Use the matrix-tree theorem to determine the number of spanning trees of the graph below.



(b) Use Kruskal's algorithm to find a spanning tree of minimum weight of the following weighted graph. Show every step of the algorithm.



Question 5 [3.5+2.5+2.5]

(a) Let G be a connected graph. Assume that G has a minimum edge-cut S and a component G_1 of $G - S$ such that every vertex in G_1 is, in G, incident with at least one edge of S. Prove that $\lambda(G) = \delta(G)$.

(b) Let G be a graph with minimum degree δ. Let H be the graph obtained from G by adding a new vertex z and edges from z to every vertex in G. Determine $\lambda(H)$. Give reasons for your answer.

(c) Let G be a k-connected graph of order n. Prove that the diameter of G cannot be greater than $1 + \frac{n-2}{k}$.

Question 6 [3+2.5]

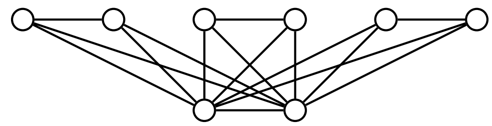
(a) Let G be a graph in which every vertex has degree 3 or 5. Prove or disprove: If $\overline{G}$, the complement G, is connected, then $\overline{G}$ is Eulerian.

(b) Let G be an Eulerian graph. Prove that no edge of G is a bridge.

Question 7 [1.5+2.5+2]

(a) Let $a \in \mathbb{N}$ with $a \geq 2$. Let G be the complete bipartite graph $K_{a,a}$. Choose three different sufficient conditions for Hamiltonian graphs from the lecture and apply them to show that G is Hamiltonian. (Do not prove that G is Hamiltonian by constructing a Hamilton cycle.)

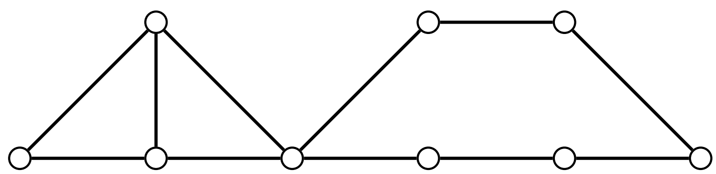
(b) Show that the following graph is NOT Hamiltonian.



(c) Given $n \in \mathbb{N}$ with $n \geq 3$. Determine the toughness of the graph $K_n - e$, the graph obtained from K_n by removing an edge.

Question 8 [2+2+2]

(a) Decide if the following graph has a strong orientation. If yes, find such an orientation.



(b) True or false? Every 1-tough graph with at least 3 vertices has a strong orientation. Give reasons.

(c) Decide if the graph in (a) has an Eulerian orientation. If no, give reasons. If yes, find such an orientation.