APM8X03: Relativity A – Final Exam

- This is an open book exam. Attempt **ALL** questions
- This paper has 130 marks available. Full Marks =130 marks
- You have 7 days to complete this exam. Submission time is 23:59.

SECTION 1: Vectors and Tensors **[30 marks]**

1. Consider the following space-time coordinate transformations:

$$T = x\sinh t, \quad X = x\cosh t$$

   1.1. Construct the parametric curves.
   Let $-2 < x < 2$ and $0 < t < 1$. (6 marks)

   1.2. Use the interval $-dT^2 + dX^2$ to extract the metric tensor. (6 marks)

2. Consider the following coordinate transformations:

$$x = \sqrt{r^2+1}\sin\theta\cos\phi, \quad y = \sqrt{r^2+1}\sin\theta\sin\phi, \quad z = r\cos\theta$$

   2.1. Plot the θ parametric curves. (4 marks)

   2.2. Plot the r parametric surfaces. (4 marks)
   Let $0 \leq r \leq 1, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq \frac{3\pi}{2}$.

3. Show that the trace of the metric tensor in an N-dimensional space is just N. *Hint: an N-dimensional space would have N coordinates, $x^1, x^2, x^3 \ldots x^N$.* (3 marks)

4. Given a tensor $G^{ij} = R^{ij} - \frac{1}{2}Rg^{ij}$, show that the trace of G^{ij} is given by

$$G = \frac{2-N}{2}R,$$

   where R is the trace of R^{ij}. (3 marks)

5. Show that if $G^{ij} = 0$ then $R^{ij} = 0$. (4 marks)

SECTION 2: Lorentz Transformations [60 marks]

1. A taxi driver approaches a red traffic light ($\nu = 4.3 \times 10^{14} Hz$), how fast would he have to be driving in order for the light to appear green ($5.7 \times 10^{14} Hz$)? Give the speed in units of c. (2 marks)

2. Peter and Carter decide to use 2 of Carter's limos to have a joust at relativistic speeds, see figure.

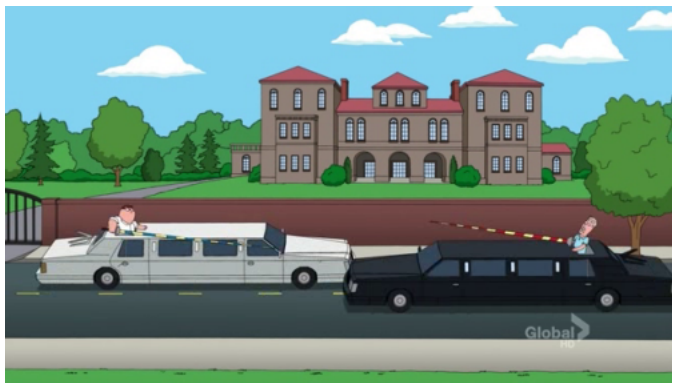

Figure 1: Peter (left) and Carter (right) about to partake in a relativistic joust.

Assuming the lances are 10 meters long and that Peter (on the left) has a limo that moves, toward the right, with a speed of $0.99c$ while Carter's limo moves at a speed of $0.999c$. See figures for more detail. Let our frame be S, the rest frame, let Carter's frame be $\bar{S}$ and Peter's be $\bar{\bar{S}}$. Assume that when the joust begins $t = \bar{t} = \bar{\bar{t}} = 0$. Answer the following questions:

2.1. What are the lengths of Peter and Carter's lances in S? (4 marks)

2.2. If the distance between Peter and Carter is 10 km at the start of the joust (in S), what distance does Peter measure to Carter? (6 marks)

2.3. Does Carter measure the same distance to Peter? (1 mark)

2.4. Explain why Peter and Carter are both confident that they cannot be struck by the other. (4 marks)

2.5. Construct the world lines of Peter and Carter (with their lances) to determine whose lance hits who first. (10 marks)

3. Consider two observers, S, and $\bar{S}$ who moves along the z-axis with speed $V = 0.8c$ relative to S. When S and $\bar{S}$ coincide $t = \bar{t} = 0$. At $t = 1$, S fires a muon (from $x = 0$) in the direction of $\bar{S}$ at a speed $v = 0.9c$ and at a time $\bar{t} = 1$, $\bar{S}$ fires a muon (from $\bar{x} = 0$) in the direction of S at a speed $v = \bar{0}.95c$.

3.1. When, according to S, does $\bar{S}$ fire their muon? (3 marks)

3.2. At what speed does $\bar{S}$ see the muon approaching? (5 marks)

3.2. What distance does the muon from $\bar{S}$ measure to S? (10 marks)

3.3. What speed does the muon from S see for the muon from $\bar{S}$? (10 marks)

3.4. Relative to $\bar{S}$, how long does the muon take to arrive at $\bar{S}$? (5 marks)

SECTION 3: Special Relativity [40 marks]

1. An electron and positron collide, each making an angle θ with the horizontal axis. Upon collision they annihilate and produce two photons (gamma rays) that move off each making an angle of φ with the horizontal axis. If both the electron and positron (the antimatter counterpart of the electron) have a speed of v and the emergent photons have a frequency ν then answer the following:

   1.1. Draw a diagram describing the collision and emerging photons. (4 marks)

   1.2. Find an expression for the frequency of the emergent photons. (4 marks)

   1.3. Find an expression for $\cos\varphi$. (2 marks)

   1.4. An observer, $\bar{S}$, moves along the horizontal axis with speed V. What angle does $\bar{S}$ see the electron make with the horizontal axis? (5 marks)

   1.5. Show that the frequency observed by $\bar{S}$ is given by

$$\bar{\nu} = \nu\Gamma(1 - \beta\cos\varphi), \text{where } \Gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \text{ and } \beta = \frac{V}{c}$$

   (5 marks)

2. The fluid stress-energy tensor: The stress energy-tensor of an ideal gas is given by

$$T^{\mu\nu} = \left[\rho(1+\epsilon) + \frac{p}{c^2}\right] u^\mu u^\nu + p\eta^{\mu\nu},$$

   where all the variables have their usual meaning.

   2.1. What is the matrix form of the stress-energy tensor in the fluid's local frame? (5 marks)

   2.2. What is the matrix form of the stress-energy tensor in the rest frame if the local speed is v along the z-axis? (5 marks)

3. Consider two charged particles aligned along the y-axis (see figure). The electromagnetic tensor in the rest frame is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{E_y}{c} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{E_y}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.1 What is the electromagnetic tensor relative to an observer moving along the z-axis with speed v? (5 marks)

3.2 Given the general structure of the electromagnetic tensor, what can you infer about magnetic fields? (5 marks)

Useful Information

$$\bar{T}^{\alpha\beta}{}_{\lambda\sigma} = \bar{x}^{\alpha}_{,\epsilon}\bar{x}^{\beta}_{,\rho}x^{\eta}_{,\lambda}x^{\phi}_{,\sigma}T^{\epsilon\rho}{}_{\eta\phi}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$x^{\mu} = \{t, x, y, z\}$$

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$$

$$u^{\mu} = \frac{dx^{\mu}}{d\tau}, \quad u^{\mu}u_{\mu} = -c^2$$

$$P^{\mu} = mu^{\mu}$$

$$A^{\mu}A_{\mu} \begin{cases} \leq 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ \geq 0 & \text{spacelike} \end{cases}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

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