APM8X03: Relativity A – Final Exam

- Attempt **ALL** questions
- This paper has 100 marks available. Full Marks =90 marks
- You have 4 hours

SECTION 1: Vectors and Tensors [35 marks]

(a) Consider the following coordinate system

$$x = a\cosh u \cos v \cos\phi, \quad y = a\cosh u \cos v \sin\phi, \quad z = a\sinh u \sin v$$

$$u \geq 0, \quad 0 \leq v \leq \pi, \quad 0 \leq \phi \leq 2\pi$$

1. Plot the ϕ-curves for $a = 1$. Let $0 < u < 1$. (4 marks)
2. Plot the v-surfaces for $a = 1$. Let $0 < u < 1$. (4 marks)
3. Find the basis vectors for the system. (3 marks)
4. Find the basis covectors for the system. (3 marks)
5. Find the metric tensor for the system. (6 marks)

(b) If F^{ij} is an antisymmetric tensor and T_{ij} is a symmetric tensor, show that the $F^{ij}Tij = 0$ for any such two tensors. (5 marks)

(c) Show that the inner product of two tensors A^{ij} and B_{ij} such that $C^i_j = A^{ik}B_{kj}$, is itself, a tensor. (5 marks).

(d) Show that, in a 2-dimensional space, the tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is traceless. *Hint: R is the trace of $R_{\mu\nu}$.* (5 marks)

SECTION 2: Lorentz Transformations **[35 marks]**

(a) Consider a frame $\bar{S}$ moving along the z-axis with speed V relative to S. If a particle in S has the 3-velocity $v = \{v_x, v_y, 0\}$, answer the following:

1. Find the components of the 3-velocity in $\bar{S}$. (5 marks)
2. Find an expression for the magnitude of the 3-velocity in $\bar{S}$ and comment on the limit as $V \to c$. (3 marks)
3. Comment on the angle that the particle moves in the $x - y$ plane relative to $\bar{S}$. (2 marks)

(b) Consider an observer S at the entrance of a 25m long garage and a second observer $\bar{S}$ driving a 100m long limousine at a speed of $0.99c$ toward the garage.

1. What length does S see for the limo? (2 marks)
2. What depth does $\bar{S}$ see for the garage? (2 marks)
3. Suppose $t = \bar{t} = 0$ when the front of the limo is exactly in line with the garage entrance. If $\bar{S}$ is right at the front of the limo, after how long does $\bar{S}$ reach the back in each of the frames? (6 marks).
4. Construct world-lines for the limo and garage (for the limo entering and leaving the garage) in each frame. (10 marks)
5. Discuss the order of the events in each frame. (5 marks)

SECTION 3: Special Relativity **[20 marks]**

1. A photon collides (head on) with a meson at rest. The meson decays into a quark and an antiquark which scatter off the line of collision with angles θ and ϕ respectively. They have velocities given by v_1 andv_2 and masses m_1 and m_2 respectively. Answer the following:

   1.1. Draw a detailed diagram describing the collision. (4 marks)

   1.2. Show that an expression for the angle between the emergent particles is given by

$$\cos(\theta + \phi) = \frac{c^2\left(m_1^2 + 2\gamma_1\gamma_2 m_2 m_1 + m_2^2\right) - m\left(c^2 m + 2h\nu\right)}{2\gamma_1\gamma_2 m_1 m_2 v_1 v_2}$$

where γ_1 and γ_2 are the respective Lorentz factors. (6 marks)

2. An observer $\bar{S}$ sets up an electric field E so that the electromagnetic tensor in the frame is given by

$$\bar{F}^{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{E_y}{c} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{E_y}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3.1 If $\bar{S}$ moves relative to S with speed V, calculate $F^{\mu\nu}$. (5 marks)

3.2 Given the general structure of the electromagnetic tensor, comment on what S observes in terms of magnetic and electric fields. (5 marks)

Useful Information

$$\bar{T}^{\alpha\beta}{}_{\lambda\sigma} = \bar{x}^{\alpha}_{,\epsilon}\bar{x}^{\beta}_{,\rho}x^{\eta}_{,\lambda}x^{\phi}_{,\sigma}T^{\epsilon\rho}{}_{\eta\phi}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$x^{\mu} = \{t, x, y, z\}$$

$$\Lambda^{\alpha}{}_{\beta} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2$$

$$u^{\mu} = \frac{dx^{\mu}}{d\tau}, \quad u^{\mu}u_{\mu} = -c^2$$

$$P^{\mu} = mu^{\mu}$$

$$A^{\mu}A_{\mu} \begin{cases} \leq 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ \geq 0 & \text{spacelike} \end{cases}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

******************* **END** *******************