Question Number	Marks Awarded
1	
2	
3	
4	
5	
6	
7	
8	
TOTAL:	

Assessor: Prof F. Nyabadza
Moderator: Dr R. Ouifki
Duration: 2 Hours
Marks: 100

UNIVERSITY OF JOHANNESBURG

APPLIED MATHEMATICS

Variational Calculus & Optimization Techniques
APM3A10
Examination: 22/06/2020

Name: ______________________ **Student Number:** ______________________

Instructions:

1. Check that this question paper consists of 3 pages in total.
2. All calculations must be shown clearly.
3. Pocket calculators are permitted.

Question 1 [13 marks]

Consider

$$I(x,y,y') = \int_0^{\pi} (e^x y' - xyy' - \frac{(y')^2}{2}) dx$$

(a) Find $y^*(x)$, the extremum, subject to the boundary conditions (10)

$$y(0) = \frac{1}{2}, y'(0) = \frac{3}{2}$$

(b) Apply the Legendre test to classify the extremum. (3)

Question 2 [10 marks]

Given that

$$I(y,y') = \int_1^3 (y-1)(y')^2 dx$$

,

(a) Show that the Euler-Langrange equation reduces to (5)

$$\frac{du}{dy} = \frac{u}{2(1-y)}.$$

(b) Use the boundary conditions (5)

$$y(1) = 2$$
$$y(3) = 1 + \sqrt[3]{9}$$

to determine the extremum of $y(x)$ for the positive constants of integration.

Question 3 [18 marks]

Given that

$$I = \int_0^L \frac{\sqrt{1+y'^2}}{h-y} dx$$

$$y(0) = y(L) = 0$$

(a) Use the Beltrami identity in (a) to show that (8)

$$y' = \sqrt{\frac{C_1 - (h-y)}{h-y}}.$$

(b) Show that the solution of (b) is a set of parametric equations (10)

$$\begin{cases} x = & \frac{K_1}{2}(\theta - \sin\theta) + K_2, \\ y = & h - \frac{K_1}{2}(1-\cos\theta) \end{cases}$$

where K_1 and K_2 are positive constants.

Question 4 [10 marks]

Use the Method of Lagrange Multipliers to find the optimal distance between the curve

$$x_2^2 + (2x_1 + 3)^2 = 1$$

and the point $(-2; 0)$ corresponding to $x_2 > 0$.

Question 5 [15 marks]

Consider the function

$$f(x) = f(x_1, x_2) = (ax_1^2 - 1)(bx_2 + 3);$$

where $a, b > \frac{1}{2}$. Given that $\nabla_x f = \begin{pmatrix} 20 \\ 3 \end{pmatrix}$ at $(x_1, x_2) = (2; 2)$.

(a) Determine a and b. (5)

(b) Hence, find both critical points of $f(x)$ (5)

(c) Classify the points in (b) by considering the eigenvalues of the Hessian H_f (5)

Question 6 [10 marks]

If the Linear Programming Problem (LPP),

$$\text{maximize } f(x) = c_1 x_1 + c_2 x_2$$

subject to

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

with $x_1 \geq 0$ $x_2 \geq 0$, and b_1 $b_2 \geq 0$ has an initial tableau

x_1	x_2	s_1	s_2	f	
$\frac{5}{2}$	1	$\frac{1}{5}$	0	0	$\frac{4}{5}$
$\frac{26}{5}$	0	$\frac{3}{5}$	1	0	$\frac{22}{5}$
$-\frac{11}{5}$	0	$\frac{5}{2}$	0	1	$\frac{8}{5}$

after one iteration of the Simplex Method, determine the various parameters in the LP.

Hence, solve the LP.

Question 7 [10 marks]

(a) Find the locations and values of the extrema of $(x, y \in \mathbb{R}^+)$ (5)

$$f(x, y) = e^{xy}$$

subject to $x^2 + y = 1$.

(b) Find the locations and values of the minima of $(x, y \in \mathbb{R})$ (5)

$$f(x, y) = \sin(x + y)$$

subject to $(x + y)^2 \leq \frac{\pi^2}{4}$.

Hint: Remember that $\cos x$ is an even function.

Question 8 [14 marks]

Maximize

$$J = \int_0^2 \left(2x - 3u - u^2\right) dt,$$

subject to $\frac{dx}{dt} = x + u$ and $x(0) = 5$ and $u \in [0, 2]$.

(a) Write down the Hamiltonian, H. (2)

(b) Show that (2)

$$u(t) = \frac{\lambda(t) - 3}{2}.$$

(c) From one of the adjoint equations, show that (5)

$$\lambda(t) = 2e^{2-t} - 1.$$

(d) Show that the optiml control u^* is given by (5)

$$u^* = \begin{cases} 2 & \text{if} \quad e^{2-t} > 4.5, \\ e^{2-t} - 2.5 & \text{if} \quad 0 < e^{2-t} \leq 4.5, \\ 0 & \text{if} \quad e^{2-t} < 2.5. \end{cases}$$

END OF QUESTION PAPER