University of Johannesburg

UNIVERSITY OF JOHANNESBURG

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE	**MAT8X08** MEASURE AND INTEGRATION THEORY
CAMPUS	**APK**
EXAM	**NOVEMBER 2019**

Date 07/11/2019 — Session 08:30 – 13:30

Assessor — Dr G Braatvedt

External Moderator — Prof B Watson

Duration 5 Hours — 50 Marks

Surname and initials:

Student number:

Tel No.:

INSTRUCTIONS:

1. The paper consists of **11** printed pages, **excluding** the front page.
2. Read the questions carefully and answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
5. **Good luck - write well :-)**

SECTION A: Theory [25]

Question 1 [6]

State each of the following:

(a) The definition of a *charge*. (2)

(b) A condition under which the order of integration can be changed for a function that is dependent on a parameter. (2)

(c) The *Hahn Decomposition Theorem*. (2)

Question 2 [7]

State and prove the *Monotone Convergence Theorem*.

Question 3 [12]

Complete the proof of the *Radon Nikodým Theorem* for finite measures:
Radon Nikodým Theorem (finite measures). Let λ and μ be finite measures defined on $\mathcal{A}$ and suppose $\lambda \ll \mu$. Then there exists a function $f \in M^+(X, \mathcal{A})$ such that

$$\lambda(E) = \int_E f \, d\mu, \qquad E \in \mathcal{A}.$$

Moreover, the function f is uniquely determined μ-almost everywhere.

For $c > 0$, let $(P(c), N(c))$ be a Hahn decomposition for the charge $\lambda - c\mu$. If $k \in \mathbb{N}$, consider the measurable sets

$$A_1 = N(c), \quad A_{k+1} = N\left((k+1)c\right) \setminus \bigcup_{j=1}^{k} A_j.$$

(a) Show, for each measurable subset E of A_k, that $(k-1)c\mu(E) \leq \lambda(E) \leq kc\mu(E)$ (3)

Define B by

$$B = X \setminus \bigcup_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} P(jc).$$

(b) Show that $\mu(B) = 0$ and infer that $\lambda(B) = 0$. (2)

Define

$$f_c(x) = \begin{cases} (k-1)c & \text{if } x \in A_k \\ 0 & \text{if } x \in B. \end{cases}$$

and we use the above to show that, for each $E \in \mathcal{A}$,

$$\int_E f_c \, d\mu \leq \lambda(E) \leq \int_E (f_c + c) \, d\mu \leq \int_E f_c \, d\mu + c\mu(X).$$

(c) Show only that $\int_E f_c \, d\mu \leq \lambda(E)$. (4)

Continuing, we arrive at the result. Finally, for uniqueness almost everywhere, suppose there is $h \in M^+$ such that

$$\lambda(E) = \int_E f \, d\mu = \int_E h \, d\mu \quad \text{for all } E \in \mathcal{A}.$$

(d) Show that $h = f$ μ-almost everywhere. (3)

SECTION B: Problems [25]

Question 4 [6]

Let $(\mathbb{R}, \mathcal{B}, \mu)$ be a measure space, where $\mathcal{B}$ denotes the Borel σ-algebra and λ denotes the Lebesgue measure. Furthermore, let

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

(a) Is f measurable? Explain. (3)

(b) Determine $\int f d\lambda$. (3)

Question 5 [4]

Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be a measure space, where $\mathcal{B}$ denotes the Borel σ-algebra and λ denotes the Lebesgue measure. If $f(x) = \left|\frac{1}{x}\right| \chi_{\mathbb{R}-\{0\}}$, determine $\int f d\lambda$.

Question 6 [4]

Prove or disprove the following independent statements:

(a) Let λ and μ be σ-finite measures defined on $\mathcal{A}$. Then for each $E \in \mathcal{A}$ there exists a function $f \in M^+(X, \mathcal{A})$ such that $\lambda(E) = \int_E f d\mu$. (2)

(b) If $\nu \ll \lambda$ and $\lambda \perp \mu$ then $\nu \perp \mu$, for measures ν, λ, μ. (2)

Question 7 [7]

Let $X = (0,\infty)$, $\mathcal{B}$ be the Borel sets of X, and λ be the Lebesgue measure. Calculate, with motivation, $\lim_n I_n$ where $f_n(x) = \dfrac{e^{-nx}}{\sqrt{x}}$ and

$$I_n = \int f_n \, d\lambda.$$

Question 8 [4]

If λ and μ are σ-finite measures on $(X, \mathcal{A})$, and $\lambda \ll \mu$, and $\mu \ll \lambda$, then show that

$$\frac{d\lambda}{d\mu} = \frac{1}{d\mu/d\lambda}, \qquad \text{almost everywhere.}$$