University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS	
MODULE	**MAT8X03** TOPOLOGY
CAMPUS	**APK**
SUPPLEMENTARY EXAM	**JANUARY 2020**
PAPER 1	**THEORY**

Examiner: Dr F Schulz
Moderator: Prof T Dube (Unisa)
Duration: 240 Minutes 70 Marks

INSTRUCTIONS:

1. The paper consists of **3** printed pages, **including** the front page.
2. Read the questions carefully and answer all questions in the provided booklets.
3. The Theory Paper is closed book.

1. Let (X, τ) be a topological space.

   (a) Define clearly what we mean by the interior $\mathrm{Int}_X(E)$ of a set E in X. (2)

   (b) Let A and B be subsets of X. Prove that $\mathrm{Int}_X(A) \cap \mathrm{Int}_X(B) = \mathrm{Int}_X(A \cap B)$. (2)

   (c) Let X be a set and let $A \mapsto A^\circ$ be a mapping from the power set $\mathscr{P}(X)$ of X into $\mathscr{P}(X)$ satisfying: (6)

      (i) $A^\circ \subseteq A$.

      (ii) $(A^\circ)^\circ = A^\circ$.

      (iii) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

      (iv) $X^\circ = X$.

   Prove that $\tau := \{G \subseteq X : G = G^\circ\}$ is a topology on X where for each $A \subseteq X$ we have $\mathrm{Int}_X(A) = A^\circ$.

Total for Question 1: 10

2. (a) Let X be a set and for each $\alpha \in A$ let X_α be a topological space. How do we define the evaluation map $e : X \to \prod_{\alpha \in A} X_\alpha$ induced by the collection $\{f_\alpha : \alpha \in A\}$ of functions $f : X \to X_\alpha$? (1)

   (b) For each $\alpha \in A$, let X_α be a topological space. Suppose that X has the weak topology induced by the collection $\{f_\alpha : \alpha \in A\}$ of functions $f : X \to X_\alpha$ and that $\{f_\alpha : \alpha \in A\}$ separates points in X. Prove that the evaluation map $e : X \to \prod_{\alpha \in A} X_\alpha$ induced by $\{f_\alpha : \alpha \in A\}$ is an embedding. (8)

Total for Question 2: 9

3. (a) Suppose that X is a first countable space and that $E \subseteq X$. Prove that $x \in \overline{E}$ if and only if there is a sequence (x_n) contained in E which converges to x. (5)

   (b) Let X be a first countable space. From part (a) it follows that $F \subseteq X$ is closed if and only if whenever $(x_n) \subseteq F$ and $x_n \to x$, then $x \in F$. Use this (or other means) to prove that $f : X \to Y$ is continuous if and only if whenever $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y. (5)

   (c) Show that the latter result in part (b) fails if we remove the assumption that X is a first countable space. Justify all of the statements in your counterexample. (4)

Total for Question 3: 14

4. (a) Define clearly what we mean by the filter generated by the net (x_λ) in X. (2)

   (b) Prove that every net has a subnet which is an ultranet. (6)

Total for Question 4: 8

5. (a) For a topological space X, the following are equivalent: (4)

   (i) X is compact.

   (ii) Each ultranet in X converges.

   (iii) Each ultrafilter in X converges.

   Show that (iii) $\Rightarrow$ (i).

   (b) Prove that the continuous image of a compact space is compact. (3)

   (c) Let $f : X \to Y$ be a continuous function between two topological spaces. Suppose that $x_\lambda \to x$ in X. Prove that $f(x_\lambda) \to f(x)$ in Y. (3)

   (d) By using part (c), show that a net (x_λ) in a product space $X := \prod_{\alpha \in A} X_\alpha$ converges to x if and only if for each $\alpha \in A$, $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ in X_α. (6)

   (e) Finally, use part (a), part (b) and part (d) to state and prove Tychonoff's Theorem. (5)

Total for Question 5: 21

6. Let X be a topological space.

   (a) Define clearly what we mean when we say that the sets H and K are mutually separated in X. (2)

   (b) Prove that a subspace E of X is connected if and only if there are no nonempty, mutually separated sets H and K in X with $E = H \cup K$. (4)

   (c) Prove that if H and K are mutually separated in X and E is a connected subset of $H \cup K$, then either $E \subseteq H$ or $E \subseteq K$. (2)

Total for Question 6: 8

Total: 70