University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE **MAT8X03**
TOPOLOGY

CAMPUS **APK**

EXAM **NOVEMBER 2019**

PAPER 1 **THEORY**

Date: 12/11/2018	Session: 08:30 – 12:30
Examiner:	Dr F Schulz
Moderator:	Prof T Dube (Unisa)
Duration: 240 Minutes	70 Marks

INSTRUCTIONS:

1. The paper consists of **3** printed pages, **including** the front page.
2. Read the questions carefully and answer all questions in the provided booklets.
3. The Theory Paper is closed book.

1. Let (X, τ) be a topological space.
   (a) Define clearly what we mean by a base for τ. (2)
   (b) Prove the following: $\mathscr{B}$ is a base for a topology on X if and only if (6)
      (i) $X = \bigcup_{B \in \mathscr{B}} B$ and
      (ii) whenever $B_1, B_2 \in \mathscr{B}$ with $p \in B_1 \cap B_2$, there is some $B_3 \in \mathscr{B}$ with
$$p \in B_3 \subseteq B_1 \cap B_2.$$

Total for Question 1: 8

2. (a) Let X be a set and for each $\alpha \in A$ let X_α be a topological space. How do we define the evaluation map $e : X \to \prod_{\alpha \in A} X_\alpha$ induced by the collection $\{f_\alpha : \alpha \in A\}$ of functions $f : X \to X_\alpha$? (1)
   (b) For each $\alpha \in A$, let X_α be a topological space. Suppose that the evaluation map $e : X \to \prod_{\alpha \in A} X_\alpha$ induced by the collection $\{f_\alpha : \alpha \in A\}$ of functions $f : X \to X_\alpha$ is an embedding. Prove that X has the weak topology induced by $\{f_\alpha : \alpha \in A\}$ and that $\{f_\alpha : \alpha \in A\}$ separates points in X. (8)

Total for Question 2: 9

3. (a) Suppose that X is a first countable space and that $E \subseteq X$. Prove that $x \in \overline{E}$ if and only if there is a sequence (x_n) contained in E which converges to x. (5)
   (b) Show that the result in part (a) may fail if we remove the assumption that X is first countable. (6)

Total for Question 3: 11

4. (a) Define clearly what we mean by a filter $\mathscr{F}$ on a set X. (3)
   (b) Prove that a filter $\mathscr{F}$ on X is an ultrafilter if and only if for each $E \subseteq X$, either $E \in \mathscr{F}$ or $X - E \in \mathscr{F}$. (6)
   (c) Prove that if a filter $\mathscr{F}$ is contained in a unique ultrafilter $\mathscr{G}$, then $\mathscr{F} = \mathscr{G}$. (4)

Total for Question 4: 13

5. (a) For a topological space X, the following are equivalent: (4)
      (i) X is compact.
      (ii) Each ultranet in X converges.
      (iii) Each ultrafilter in X converges.

   Show that (iii) $\Rightarrow$ (i).
   (b) Prove that the continuous image of a compact space is compact. (3)
   (c) Let $f : X \to Y$ be a continuous function between two topological spaces. Suppose that $x_\lambda \to x$ in X. Prove that $f(x_\lambda) \to f(x)$ in Y. (3)
   (d) By using part (c), show that a net (x_λ) in a product space $X := \prod_{\alpha \in A} X_\alpha$ converges to x if and only if for each $\alpha \in A$, $\pi_\alpha(x_\lambda) \to \pi_\alpha(x)$ in X_α. (6)

(e) Finally, use part (a), part (b) and part (d) to state and prove Tychonoff's Theorem. (5)

Total for Question 5: 21

6. (a) Define clearly what we mean when we say that a topological space X is disconnected. (2)

(b) Show that the continuous image of a connected space is connected. (2)

(c) Prove that if $X = \bigcup_{\alpha \in A} X_\alpha$ where each X_α is connected and $\bigcap_{\alpha \in A} X_\alpha \neq \varnothing$, then X is connected. (4)

Total for Question 6: 8

Total: 70