University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE	**MAT8X03** TOPOLOGY
CAMPUS	**APK**
EXAM	**NOVEMBER 2019**
PAPER 2	**PROBLEMS**

Date: 14/11/2018 — Session: 08:30 – 14:30

Examiner: Dr F Schulz

Moderator: Prof T Dube (Unisa)

Duration: 360 Minutes — 40 Marks

INSTRUCTIONS:

1. The paper consists of **2** printed pages, **including** the front page.
2. Read the questions carefully and answer all questions in the provided booklets.
3. You are allowed a copy of the book *General Topology* by Stephen Willard.

1. Let $\mathbb{Z}$ be the set of integers and let $p > 0$ be a prime integer. Define $d_p : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by $d_p(m,n) = 0$ when $m = n$; otherwise, $d_p(m,n) = 1/p^r$, where p^r is the largest non-negative integer power of p which divides $m - n$.

   (a) Prove that $(\mathbb{Z}, d_p)$ is a metric space. (5)

   (b) Describe the 1-disk centered at 0 in $(\mathbb{Z}, d_p)$. (1)

Total for Question 1: 6

2. We introduce a topology on the real line $\mathbb{R}$ as follows: a set is open if and only if it is of the form $U \cup V$ where U is an open subset of $\mathbb{R}$ with the usual topology and V is any subset of the irrationals. Call the resulting space $\mathbf{S}$, (the *scattered line*).

   (a) Verify that the description of an "open set" above indeed yields a topology on $\mathbf{S}$. (4)

   (b) Describe an efficient nhood base at (4)

      (i) the rational points

      (ii) the irrational points

      in $\mathbf{S}$.

   (c) Use part (b) to describe a base for $\mathbf{S}$. (1)

   (d) Let $\mathbb{Q}$ be the set of rational numbers. Find $\mathrm{Cl}_{\mathbf{S}}(\mathbb{Q})$. (1)

Total for Question 2: 10

3. Exhibit topological spaces X, Y and Z such that $X \times Y$ is homeomorphic to $X \times Z$, but Y is not homeomorphic to Z. (4)

4. Suppose that X_α is a topological space for each $\alpha \in A$ and that $X := \prod_{\alpha \in A} X_\alpha$ is endowed with the Tychonoff topology. For each $\alpha \in A$ let $\emptyset \neq F_\alpha \subseteq X_\alpha$. Prove that (5)

$$\mathrm{Cl}_X \left(\prod_{\alpha \in A} F_\alpha \right) = \prod_{\alpha \in A} \mathrm{Cl}_{X_\alpha}(F_\alpha).$$

5. (a) If $(x_\lambda)_{\lambda \in \Lambda}$ is a net in $\prod_{\alpha \in A} X_\alpha$ which clusters at a point x prove that for each $\alpha \in A$, the net $(\pi_\alpha(x_\lambda))_{\lambda \in \Lambda}$ clusters at $\pi_\alpha(x)$ in X_α. (4)

   (b) Show that the converse of part (a) fails even in $\mathbb{R} \times \mathbb{R}$. (4)

Total for Question 5: 8

6. Let X and Y be topological spaces with Y compact. If $f : X \to Y$ is closed, onto and has the property that $f^{-1}(\{y\})$ is a compact subspace of X for each $y \in Y$, prove that X is compact. (5)

7. Prove that the circle $\mathbb{S}^1$ is not homeomorphic to any subspace of $\mathbb{R}$ with the usual topology. (4)

Total: 42