University of Johannesburg

FACULTY OF SCIENCE

Pure and Applied Mathematics (APK)
MAT02B3
Introductory Abstract Algebra
SUPPLEMEMTARY EXAM
January 2020

Examiner .... Prof P. Dankelmann
External Examiner ... Dr. M.J. Morgan (University of KwaZulu-Natal)
Time .... 3 hours
Total marks 38.5
Student number ....
Cell number ....

Please read the following instructions carefully:

1) Answer all questions.
2) Questions are to be answered in the exam books provided.
3) This paper consists of this cover page and two pages of questions.
4) Neither books, nor notes, nor calculators are to be used.

Abstract Algebra MAT02B3

Supplementary Exam - January 2020

Marks: 38.5 ....total time 3 hours

Question 1 [0.5+2+2+2]

(a) Define what is meant by a group.
(b) Let G be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^3 \end{pmatrix}$$

where $a \in \mathbb{R} - \{0\}$. Is G a group under matrix multiplication? Prove your answer. (You may use the fact that matrix multiplication is associative, without proving it.)
(c) Let $G = \{0, 2, -2, 4, -4, 6, -6, \ldots\}$, i.e., G is the set of all even integers. Define a binary operation $*$ on G by

$$a * b = \frac{1}{2}ab,$$

where ab stands for the usual product of a and b. Decide if G is a group under the operation $*$. Prove your answer.
(d) Let G be a group and $a \in G$. Prove that a has only one inverse.

Question 2 [0.5+2.5]

(a) State the subgroup test for finite groups. (No proof is required.)
(b) Let G be an Abelian group and let H and K be subgroups of G. Prove that the set

$$HK = \{hk \mid h \in H,\ k \in K\}$$

is a subgroup of G.

Question 3 [0.5+1.5+1]

(a) Define what is meant by a generator of a group and by a cyclic group.
(b) Let a be an element of order n of a group G. Let $k \in \mathbb{N}$ with $\gcd(n, k) = 1$. Prove: $a \in \langle a^k \rangle$.
(c) Determine all generators of $\mathbb{Z}_{12}$. Explain how you arrive at your answer!

Question 4 [0.5+1+0.5+1]

(a) Let α be the permutation of $\{1, 2, 3, 4, 5, 6\}$ defined by

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 3 & 1 & 5 \end{bmatrix}.$$

Write α as a product of disjoint cycles.
(b) Let $\alpha = (23)(13)(24)$. Determine the order of α. Use your result to determine α^{1002}.
(c) Define what is meant by an even permutation.
(d) True or false? Let $n \geq 2$. If $\alpha, \beta \in S_n$ are two odd permutations, then the product $\alpha\beta$ is an odd permutation. Give reasons.

Question 5 [0.5+2+1.5]

(a) Let G and $\overline{G}$ be groups. Define what is meant by an isomorphism from G to $\overline{G}$.
(b) Let G and $\overline{G}$ be groups and let Φ be an isomorphism from G to $\overline{G}$. Prove: If e_G is the identity of G and $e_{\overline{G}}$ is the identity of $\overline{G}$, then $\Phi(e_G) = e_{\overline{G}}$.
(c) Determine all automorphisms of the group $\mathbb{Z}_{12}$ under addition. Give a reason why there are not more automorphisms.

Question 6 [0.5+1+0.5+1.5+1.5]

(a) Let G be a group, let $H \leq G$ and $a \in G$. Define what is meant by the left coset of H containing a.

(b) Let $G = U(9)$ and let H be the subgroup $\{1, 8\}$ of G. How many different cosets does H have in G? For each of these different cosets list their elements.

(c) State Lagrange's Theorem. (You don't need to prove Lagrange's Theorem.)

(d) Use Lagrange's Theorem to prove that every group of prime order is cyclic.

(e) Let G be the group $\{(1), (124), (142), (35), (124)(35), (142)(35)\}$ of permutations of the set $S = \{1, 2, 3, 4, 5\}$. Determine $\mathrm{orb}_G(4)$ and $\mathrm{stab}_G(4)$, and verify that for these two sets the Orbit-Stabiliser Theorem holds.

Question 7 [2+1.5]

(a) Determine all elements of order 6 in the direct product $\mathbb{Z}_6 \oplus \mathbb{Z}_3$. Explain how you arrive at your answer.

(b) True or false: every Abelian group of order 20 contains an element of order 4. Give reasons!

Question 8 [0.5+2+1]

(a) Let G be a group and H a normal subgroup of G. What are the elements of the factor group G/H and how is the operation on G/H defined?

(b) Consider the group $G = U(20)$ and the normal subgroup $H = \{1, 9\}$. How many elements does the factor group G/H have? Which element is the identity of G/H? Choose an element of G/H which is not the identity, and determine its order.

(c) True or false: If G is an Abelian group and H a normal subgroup of G, then the factor group G/H is Abelian? Give reasons.

Question 9 [0.5+2+1]

(a) Let G and $\overline{G}$ be groups. Define what is meant by a homomorphism from G to $\overline{G}$, and by the kernel of a homomorphism.

(b) Let $\Phi : G \to \overline{G}$ be a homomorphism. Use the first isomorphism theorem and Lagrange's theorem to show that $|\Phi(G)|$ divides both $|G|$ and $|\overline{G}|$.

(c) Determine all homomorphic images of $\mathbb{Z}_{11}$ (up to isomorphism). Explain your answer.

Question 10 [1+0.5+2]

(a) Let R be the ring of all 2×2 matrices with real entries under matrix addition and matrix multiplication. Let

$$S = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R}\}.$$

Show that S is a subring of R.

(b) Let R be a ring. Define what is meant by a unity of R.

(c) Prove that every finite integral domain is a field.