University of Johannesburg

FACULTY OF SCIENCE

Pure and Applied Mathematics (APK)
MAT02B3
Introductory Abstract Algebra
EXAM
November 2019

Examiner .... Prof P. Dankelmann
External Examiner ... Dr. M.J. Morgan (University of KwaZulu-Natal)
Time .... 3 hours
Total marks 37.5
Student number ....
Cell number ....

Please read the following instructions carefully:

1) Answer all questions.
2) Questions are to be answered in the exam books provided.
3) This paper consists of this cover page and two pages of questions.
4) Neither books, nor notes, nor calculators are to be used.

Abstract Algebra MAT02B3

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Question 1 [0.5+2+1.5+2]

(a) Define what is meant by a group.
(b) Let G be the set of all invertible 2×2 matrices over the real numbers. Define a binary operation $*$ on G by

$$A * B = BA$$

where BA means the usual matrix multiplication. Is G a group under the operation $*$? Prove your answer. (You may use the fact that matrix multiplication is associative without proving it.)
(c) Let $G = \{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\}$. Prove that G is not a group under multiplication.
(d) Let G be a group. Prove that G has only one identity element.

Question 2 [0.5+2]

(a) Let G be a group and $S \subseteq G$. Describe briefly (and without proof) either the one-step subgroup test or the two-step subgroup test for S being a subgroup of G.
(b) Let $GL(2, \mathbb{R})$ be the group of all real 2×2-matrices under multiplication. Let H be the subset of $GL(2, \mathbb{R})$ containing the 2×2-matrices A with

$$\det(A) \in \mathbb{Q}^{>0},$$

where $\mathbb{Q}^{>0}$ is the set of all positive rational numbers. Is H a subgroup of $GL(2, \mathbb{R})$? Prove your answer.

Question 3 [0.5+1.5+1]

(a) State the fundamental theorem on cyclic groups (no proof is required).
(b) Let a be an element of a group G. Recall that $\langle a \rangle = \{a^k \mid k \in \mathbb{Z}\}$. Prove: If a has order n, then $\langle a \rangle = \{e, a, a^2, a^3, \ldots, a^{n-1}\}$.
(c) Determine all generators of $\mathbb{Z}_9$. Explain how you arrive at your answer!

Question 4 [0.5+1+1]

(a) Let α be the permutation of $\{1, 2, 3, 4, 5, 6, 7\}$ defined by

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 5 & 4 & 3 & 2 \end{bmatrix}.$$

Write α as a product of disjoint cycles.
(b) Let $\alpha = (14)(134)$ Determine the order of α. Use your result to determine α^{2002}.
(c) Determine for each of the following permutations if it is odd or even: $\alpha = (321)$, $\beta = (13)(32)(24)(41)$. Give reasons for your answers.

Question 5 [0.5+2+1.5]

(a) Let G be a group. Define what is meant by an automorphism of G.
(b) Let G and $\overline{G}$ be groups and let Φ be an isomorphism from G to $\overline{G}$. Prove: If a is an element of G and a^{-1} its inverse in G, then $\Phi(a^{-1})$ is the inverse of $\Phi(a)$ in $\overline{G}$.
(c) For each of the following pairs of groups either give an isomorphism (you do not need to prove that it is an isomorphism) or explain why there is no isomorphism.
(i) $\mathbb{Z}_{12}$ and $\mathbb{Z}_6 \oplus \mathbb{Z}_2$,
(ii) $\mathbb{Z}$ under addition, and $2\mathbb{Z}$ (the set of all even integers) under addition,
(iii) $U(n)$ and $\mathrm{Aut}(\mathbb{Z}_n)$.

Question 6 [1+2.5+1.5+1.5]

(a) Let $G = U(11)$ and let H be the subgroup $\{1, 10\}$ of G. How many different left cosets does H have in G? For each of these different left cosets list their elements.
(b) Prove Lagrange's Theorem: If G is a finite group and H a subgroup of G, then $|H|$ divides $|G|$. (You may use the following lemma without proving it: If aH and bH are two cosets of H in G, then (i) aH and bH are either equal or disjoint, and (ii) $|aH| = |bH|$.)
(c) Let G be a group of order n, let e be the identity of G, and let $a \in G$. Prove that $a^n = e$.
(d) Let G be the group $\{(1), (125), (152), (34), (125)(34), (152)(34)\}$ of permutations of the set $S = \{1, 2, 3, 4, 5\}$. Determine $\mathrm{orb}_G(2)$ and $\mathrm{stab}_G(2)$, and verify that for these two sets the Orbit-Stabiliser Theorem holds.

Question 7 [1.5+1.5]

(a) Determine the number of elements of order 6 in the direct product $\mathbb{Z}_3 \oplus \mathbb{Z}_{10}$.
(b) True or false: every Abelian group of order 28 has an element of order 2. Give reasons!

Question 8 [0.5+2]

(a) Let G be a group and H a subgroup of G. Define what is meant by H being a normal subgroup of G.
(b) Consider the group $G = \mathbb{Z}_{20}$ and the normal subgroup $H = \{0, 4, 8, 12, 16\}$. What is the order of the factor group G/H? What is the identity of G/H? Choose an element of G/H which is not the identity, and determine is order. (No proof is required.)

Question 9 [2+0.5]

(a) For each of the following mappings between two groups say whether it is a homomorphism or not (no proof required). For those mappings that are homomorphisms determine the kernel.

(i) $\Phi_1 : GL(2, \mathbb{R}) \to \mathbb{R} - \{0\}$ with $\Phi_1(A) = \det(A)$ (where the operation in $\mathbb{R} - \{0\}$ is multiplication).
(ii) $\Phi_2 : \mathbb{R} \to \mathbb{R}$ with $\Phi_2(x) = \frac{1}{2}x$ (where the operation in $\mathbb{R}$ is addition).

(b) State the First Isomorphism Theorem. (No proof is required.)

Question 10 [1+0.5+2+1.5]

(a) Let $\mathbb{C}$ be the ring of all complex numbers under addition and multiplication. Consider the subset S, defined by

$$S = \{a + bi \mid a, b \in \mathbb{Z}\}.$$

Show that S is a subring of R.
(b) Let R be a ring. Define what is meant by a unit of R.
(c) Let R be an integral domain and let $a, b, c \in R$ with $a \neq 0$ and $ab = ac$. Prove that $b = c$.
(d) Let $n \in \mathbb{N}$. Prove that every non-zero element of $\mathbb{Z}_n$ is either a zero-divisor or a unit.