University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS	
MODULE	**MAT2B20/MAT02B2** LINEAR ALGEBRA 2B
CAMPUS	**APK**
SUPPLEMENTARY EXAM	**JANUARY 2020**

Examiner	Dr F Schulz
Moderator	Dr G Braatvedt
Duration 150 Minutes	50 Marks

Surname and initials: _____

Student number: _____

Identity number: _____

INSTRUCTIONS:

1. The paper consists of **10** printed pages, **excluding** the front page.
2. Read the questions carefully and answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
5. Non-programmable calculators are allowed.

Question 1 [4]

Choose the correct option for the multiple choice questions below and write your answer in the table provided.

Question	Answer
1.1	
1.2	
1.3	
1.4	

(1.1) If 3 is an eigenvalue of the matrix A, which value is guaranteed to be an eigenvalue of the matrix A^3? (1)

(a) 3 (b) 27 (c) 9 (d) -3 (e) 0

(1.2) Let $\overline{p} = 4 - x$ and $\overline{q} = 4x^2 - 1$ be vectors in $\mathcal{P}_2$. Find the cosine of the angle between $\overline{p}$ and $\overline{q}$ when $\mathcal{P}_2$ is equipped with the evaluation inner product at $-\frac{1}{2}$ and 4. (1)

(a) 1 (b) 0 (c) -1 (d) $\frac{1}{2}$ (e) $-\frac{\sqrt{3}}{2}$

(1.3) Consider $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Is A an orthogonal matrix? (1)

(a) Yes (b) No

(1.4) Suppose that V and W are vector spaces with dimensions 7 and 5, respectively. If $T : V \to W$ is a linear transformation having nullity 5, what is rank(T)? (1)

(a) 0 (b) 2 (c) 4 (d) 3 (e) 5

Question 2 [6]

Determine whether the following statements are TRUE or FALSE. Motivate the statement if TRUE; provide a counter-example if FALSE.

(a) If the characteristic polynomial of A does not have a constant term, then A is not invertible. (2)

(b) If $A\overline{x} = \overline{b}$ is an inconsistent linear system, then $A^T A\overline{x} = A^T\overline{b}$ is also inconsistent. (2)

(c) If $T : V \to V$ is a linear transformation and V is infinite-dimensional, then the rank of T is infinite. (2)

Question 3 [5]

Suppose the characteristic polynomial of some matrix A is found to be

$$p(\lambda) = \lambda(\lambda + 6)^2(\lambda + 3)^3.$$

In each part, answer the question and justify your answer.

(a) Is the rank of A equal to 6? (2)

(b) When will A be diagonalizable? (2)

(c) What is the value of $\det(A)$? (1)

Question 4 [4]

Let A be an $n \times n$ matrix. We say that A is an idempotent matrix if $A^2 = A$.

(a) Give an example of a nonzero, nonidentity idempotent matrix. (1)

(b) Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of any idempotent matrix A. (3)

Question 5 [4]

Prove the Cauchy-Schwarz Inequality for nonzero vectors; that is, prove that if $\overline{u} \neq \overline{0}$ and $\overline{v} \neq \overline{0}$ are vectors in an inner product space V, then

$$|\langle \overline{u}, \overline{v} \rangle| \leq \|\overline{u}\| \|\overline{v}\|.$$

Question 6 [3]

Let M_{22} be the inner product space with the inner product defined by:

$$\langle U, V \rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3 + 3u_4 v_4 \quad ; \quad \text{for } U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \text{ and } V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \text{ in } M_{22}.$$

Find a basis for $W^{\perp}$ if W is the subspace of symmetric matrices.

Question 7 [5]

(a) Let V be a real inner product space. For vectors $\overline{u}$ and $\overline{v}$ in V suppose that

$$\|\overline{u}+\overline{v}\|^2 = \|\overline{u}\|^2 + \|\overline{v}\|^2 .$$

Show that $\overline{u}$ and $\overline{v}$ are orthogonal vectors. (2)

(b) Use part (a) to prove that if an $n \times n$ matrix A satisfies $\|A\overline{x}\| = \|x\|$ for all $\overline{x} \in \mathbb{R}^n$, then A is an orthogonal matrix. (3)

Question 8 [5]

If $b \neq 0$, orthogonally diagonlize $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

Question 9 [3]

Consider the quadratic form $2x^2 - 4y^2 + 2xy = 1$.

(a) Determine whether the quadratic form is positive definite, negative definite, or indefinite. Show all calculations. (2)

(b) Hence, is this conic section a hyperbola, ellipse or neither? Explain. (1)

Question 10 [3]

Prove or disprove the statement: If A is an $n \times n$ symmetric matrix such that the determinant of every principal submatrix is negative, then A is negative definite.

Question 11 [3]

Consider the theorem:

Every real n-dimensional vector space V is isomorphic to $\mathbb{R}^n$.

(a) Define the transformation $T : V \to \mathbb{R}^n$ used to prove this theorem. (1)

(b) Prove that T (defined in (a)) is linear. (2)

Question 12 [5]

Let $T : \mathcal{P}_2 \to \mathbb{R}^3$ be defined by

$$T(\overline{p}) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} \quad ; \quad \overline{p} \in \mathcal{P}_2.$$

(a) Determine the kernel of T. (2)

(b) Hence, is T onto $\mathbb{R}^3$? Explain. (1)

(c) Is T invertible? If so, find a formula for T^{-1}. If not, explain why this is not the case. (2)