UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE	**MAT2B10/MAT01B2** **MULTIVARIABLE AND VECTOR CALCULUS**
CAMPUS	**APK**
ASSESSMENT	**MAIN EXAMINATION**

DATE :	**20 NOVEMBER 2019**
ASSESSOR(S):	**DR C. RATHILAL** **MR M. SIAS**
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DURATION : 120 MINUTES	**MAXIMUM MARKS : 50**

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

CONTACT NUMBER ______________________________

NUMBER OF PAGES: 1 + 13 PAGES

INSTRUCTIONS:
1. **ANSWER ALL THE QUESTIONS ON THE PAPER IN PEN.**
2. **CALCULATORS ARE ALLOWED.**
3. **SHOW ALL CALCULATIONS AND MOTIVATE ALL ANSWERS.**
4. **IF YOU REQUIRE EXTRA SPACE, CONTINUE ON THE ADJACENT BLANK PAGE AND INDICATE THIS CLEARLY.**

Question 1 [7]

For questions (1.1) - (1.7), please circle only **ONE** correct answer:

(1.1) Find the following limit, if it exists:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{8xy + |z|}{\sqrt{x^2 + y^2 + z^2}}$$

(a) 0

(b) $\frac{1}{\sqrt{2}}$

(c) 8

(d) $\frac{8}{\sqrt{2}}$

(e) The limit does not exist.

(1.2) If $f_x(a, b)$ and $f_y(a, b)$ both exist, then f is differentiable at (a, b).

(a) True

(b) False

(1.3) If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$curl\ (\mathbf{F} \bullet \mathbf{G}) = curl\ \mathbf{F} \bullet curl\ \mathbf{G}$$

.

(a) True

(b) False

(1.4) Find the Jacobian of the transformation $x = 5\alpha \sin\beta$ and $y = 4\alpha \cos\beta$.

(a) 9α

(b) $-20\alpha \sin\beta \cos\beta$

(c) -20α

(d) $-\alpha$

(e) 36α

(1.5) Find all the saddle points of the function $f(x, y) = x \sin \frac{y}{3}$.

(a) $(0, 3\pi n)$ where $n \in \mathbb{Z}$

(b) $(0, \frac{\pi n}{3})$ where $n \in \mathbb{Z}$

(c) $(3\pi n, 1)$ where $n \in \mathbb{Z}$

(d) $(\frac{3n}{\pi}, 0)$ where $n \in \mathbb{Z}$

(e) $(3\pi n, 0)$ where $n \in \mathbb{Z}$

(1.6) $\displaystyle\int_{-1}^{1} \int_{0}^{1} e^{x^2+y^2} \sin y \; dx \; dy = 0.$

(a) True

(b) False

(1.7) $\displaystyle\int_{-C} f(x, y) \; ds = -\int_{C} f(x, y) \; ds.$

(a) True

(b) False

Question 2 [6]

(2.1) State the precise definition of the limit of a function of two variables. (2)

(2.2) Use the precise definition of the limit to show that

$$\lim_{(x,y)\to(0,0)} \frac{2x^3 + y^3}{x^2 + y^2} = 0$$

(4)

Question 3 [5]

Suppose that f is a differentiable function of x and y. Prove that f has a directional derivative in the direction of any unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and

$$\mathrm{D}_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Question 4 [5]

(4.1) Maximize $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + y + z = k$, where k is a constant and x, y and z are all positive. (3)

(4.2) Use your answer in (4.1) to prove that

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$$

for all positive real numbers x, y and z. (2)

Question 5 [7]

(5.1) Evaluate $\iiint_E x^2 dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$. (4)

(5.2) Set up the triple integral to determine the volume of the solid region that lies above the cone $z = \sqrt{x^2 + y^2}$, and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Clearly show all calculations and/or diagrams to justify your answer. (3)

Question 6 [4]

Evaluate the integral $\iint_E x^2 dA$, where E is the region bounded by the ellipse $9x^2 + 4y^2 = 36$, using the following transformation; $x = 2u$ and $y = 3v$.

Question 7 [4]

Using Green's Theorem, evaluate the line integral $\oint_C (y - \cos x)\, dx + \sin x\, dy$, where C is the triangle with vertices $(0, 0), (\frac{\pi}{2}, 0), (\frac{\pi}{2}, 1)$, followed in the anticlockwise direction.

Question 8 [9]

Consider the vector field $\mathbf{F}(x, y) = (e^x \sin y + 2y, e^x \cos y + 2x - 2y)$, $(x, y) \in \mathbb{R}$.

(8.1) Show that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^2$. (1)

(8.2) Find the potential function f of $\mathbf{F}$. (3)

(8.3) Evaluate the integral $\int_C (e^x \sin y + 2y)\, dx + (e^x \cos y + 2x - 2y)\, dy$, where C is the smooth curve from $(1, 0)$ to $(2, \pi)$. (2)

(8.4) Evaluate the integral $\oint_C (e^x \sin y + 3y)\, dx + (e^x \cos y + 2x - 2y)\, dy$, where C is the unit circle $x^2 + y^2 = 1$ oriented clockwise. (3)

Question 9 [3]

Consider the following Theorem:

If f is a function of three variables that has continuous second order partial derivatives, then $curl(\nabla f) = \boldsymbol{0}$.

Using the above Theorem, show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ is not conservative.