UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MAT0AB2/MATEAB2

ENGINEERING MATHEMATICS 0AB2/2B2

EXAM

11 NOVEMBER 2019

EXAMINERS: Dr. J. Mba
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INTERNAL MODERATOR: Dr. E. Joubert

TIME: **120** MINUTES **50** MARKS

SURNAME AND INITIALS: ______________________________

STUDENT NUMBER: ______________________________

TEL NO.: ______________________________

INSTRUCTIONS:

1. The paper consists of **11** printed pages, **excluding** the front page.
2. Answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Read the questions carefully.
5. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
6. **No calculators are allowed.**
7. Good luck!

Question 1 [12]

Determine whether the following statements are TRUE or FALSE. Motivate the statement if TRUE; provide a counterexample if FALSE.

(a) The only $n \times n$ matrix A such that $\operatorname{rank}(A) = n$ is the identity matrix. (2)

TRUE	
FALSE	

(b) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\ker(T) = \{\overline{0}\}$, then $[T]$ is invertible. (2)

TRUE	
FALSE	

(c) Suppose that $\{\overline{x}, \overline{y}\}$ is a set of linearly independent eigenvectors of A. Then $\overline{x} + \overline{y}$ is also an eigenvector of A. (2)

TRUE	
FALSE	

(d) If $\overline{u}$ and $\overline{v}$ are vectors in $W^{\perp}$, then $\overline{u}+\overline{v} \in W^{\perp}$. (2)

TRUE	
FALSE	

(e) If A is a $n \times n$ matrix with $\det(A) \neq 0$, then A has a QR-decomposition. (2)

TRUE	
FALSE	

(f) If $A\overline{x} = \overline{b}$ is an inconsistent linear system, then $A^T A\overline{x} = A^T\overline{b}$ is also inconsistent. (2)

TRUE	
FALSE	

Question 2 [3]

Find the dimension of the subspace of M_{nn} consisting of all diagonal $n \times n$ matrices. Motivate your answer clearly.

Question 3 [5]

Let $S = \{\overline{e}_1, \overline{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $B = \{\overline{v}_1, \overline{v}_2\}$ be the basis that results when the vectors in S are rotated counter-clockwise by an angle of θ radians.

(a) Find the transition matrix $P_{B\to S}$. (3)

(b) Let $P = P_{B\to S}$ and show that $P^T = P_{S\to B}$. (2)

Question 4 [5]

Let

$$A = \begin{bmatrix} 3 & -1 \\ -8 & 3 \end{bmatrix}.$$

(a) Express A as a product of elementary matrices, and then describe the geometric effect of multiplying by A in $\mathbb{R}^2$ in terms of shears, compressions, expansions and reflections. (4)

(b) What is the rank of A? Explain. (1)

Question 5 [3]

Prove or disprove: If A is a 3×3 diagonalizable matrix with (not necessarily distinct) eigenvalues λ_1, λ_2 and λ_3, then $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$.

Question 6 [3]

Find all complex scalars k, if any, for which $\mathbf{u}$ and $\mathbf{v}$ are orthogonal in $\mathbb{C}^3$ if

$$\mathbf{u} = (3i, 2+i, i) \text{ and } \mathbf{v} = (2i, 1+3i, k)\,.$$

Question 7 [6]

Consider the linear differential equation $y'' - y' - 2y = 0$

(a) Show that the substitution $y_1 = y$ and $y_2 = y'$ lead to the system (2)
$y_1' = y_2$
$y_2' = 2y_1 + y_2$

(b) Solve the system obtained in (a). (3)

(c) Use the result obtained in (b) to solve the original linear differential equation. (1)

Question 8 [3]

Let $\mathcal{C}[-1,1]$ be the vector space of continuous functions over the interval $[-1,1]$. Assume that $\mathcal{C}[-1,1]$ has the following inner product

$$\langle \overline{f}, \overline{g} \rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

Let $\overline{f} = f(x) = x^2 - x$ and $\overline{g} = g(x) = x + 1$. Find the cosine of the angle between $\overline{f}$ and $\overline{g}$.

Question 9 [2]

Let $\overline{u} = (1, -6, 1)$, $\overline{v}_1 = (-1, 2, 1)$ and $\overline{v}_2 = (2, 2, 4)$.
Find the orthogonal projection of $\overline{u}$ on the subspace of $\mathbb{R}^3$ spanned by vectors $\overline{v}_1$ and $\overline{v}_2$.

Question 10 [5]

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find a QR-decomposition of A

Question 11 [3]

Let $A = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}$. Find a unitary matrix P that diagonalizes the matrix A.