APM8X13: Numerical Analysis B Final Exam

Total: 100 Marks

1 Convection Equation: 35 Marks

The equation for thermal convection is given by

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}(Tu) + \frac{\partial}{\partial y}(Tv) = \nu \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right),$$

where u and v make up a velocity field.

1. Discuss the problem arising when using the FTCS method for the equation. (5 marks)

2. Comparing the different dynamics in the equation, make a statement about the size of the time step required if an explicit method is implemented. (10 marks)

Now consider the case of a circulating flow $u = y$ and $v = -x$ and the intention of using the Lax-Friedrichs scheme for the hyperbolic part and the BTCS method for the parabolic part, given the information below:

$$\nu = 1, \;\; x, y \in [-1, 1]$$

$$T(0,x,y) = \begin{cases} 1, & y = -1 \\ 0 & everywhere\ else \end{cases}, \quad T(t,-1,y) = T(t,1,y) = T(t,x,1) = 0 \quad T(t,x,-1) = 1.$$

$\delta x = \delta y = 0.2$

3. Discuss the method of numerical solution making careful mention of 1) initialising the problem and implementing the methods 2) the appropriate time step and 3) give some indication of what you would expect of the solution to look like (no plot needed). (20 marks)

2 Shallow Water Equations: 65 Marks

Consider the Shallow Water Equations given in the attached paper.

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0$$

$$\frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2} g h^2 \right) = -gh\frac{\partial b}{\partial x}$$

1. Discuss the idea of consistency in the context of a numerical flux. (5 marks)

2. Determine the characteristic waves arising from the system of equations governing shallow water flow: you may ignore the source term. (10 marks)

3. Look at section 4.4 of the paper and use the local Lax-Friedrichs method to solve the system of equations: use 100 cells (not 500 as in the paper) together with Neumann boundary conditions on both ends − take $g = 9.8$ and $CFL = 0.9$. (50 marks)

   Hint: use a central difference to approximate the derivative on the right hand side.