Question Number	Marks Awarded
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL:	

Assessor: Prof F. Nyabadza
Moderator: Dr R. Ouifki
Duration: 3 Hours
Marks: 100

APPLIED MATHEMATICS

Partial Differential Equations
APM8X11
Supplementary Examination: 8/01/2020

Name: ______________________ **Student Number:** ______________________

Instructions:

1. Check that this question paper consists of 20 pages in total.
2. All calculations/working must be shown in the spaces provided. Additional working space is provided on the next page.
3. Pocket calculators are permitted.
4. The assessment policies of the university are applicable.
5. Books or notes are **not** allowed.
6. Page20 will **not** be marked. It is reserved for rough work.

Question 1 [4 marks]

Classify each of the following equations as elliptic, hyperbolic or parabolic, clearly justifying your reasoning.

(a) (2)

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial y^2}.$$

(b) (2)

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + 3y.$$

Solution

Question 2 [4 marks]

(a) Show that $v = G(3y + x)$, where G is an arbitrary differentiable function, is a general solution of the equation (2)

$$-3\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

(b) Find the particular solution which satisfies (2)

$$v(x, 0,) = 2\sin x.$$

Solution

Working space

Question 3 [4 marks]

Find the general solution of

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 2u.$$

Solution

Question 4 [10 marks]

(a) Solve the equation (5)

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y.$$

(b) Find the particular solution for which (5)

$$z(x, 0) = x^2, \quad v(1, y) = \cos y.$$

<u>*Solution*</u>

Working space

Question 5 [17 marks]

For the function

$$f(x) = x^2, \qquad \text{on} \;\; 0 \le x \le 2\pi;$$

(a) Sketch the graph of $f(x)$ on the expanded interval $-4\pi \le x \le 4\pi$ given that the function is periodic. (2)

(b) Find the corresponding Fourier series. (10)

(c) Prove that (5)

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots = \frac{\pi^2}{12}.$$

Solution

Working space

Question 6 [10 marks]

Expand

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \\ \pi - x, & 0 < x < \pi, \end{cases}$$

in a Fourier series.

Solution

Working space

Question 7 [7 marks]

Show that the function

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}}$$

solves the diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}.$$

Solution

Working space

Question 8 [9 marks]

Given that $u(x,t)$ is the density of some coloured molecules at time t and state space x, the movement and spread of the particles in a column of length L is described by the equation

$$\frac{\partial u}{dt} = ru + D\frac{\partial^2 u}{dx^2}, \qquad 0 < x < L,\ r > 0, \quad D > 0,$$

with $u(0,t) = u(L,t) = 0$ and $u(x,0) = u_0(x)$.

(a) By letting $u(x,t) = e^{rt}N(x,t)$, show that the system reduces to (5)

$$\frac{\partial N}{dt} = D\frac{\partial^2 N}{dx^2},$$

with $N(0,t) = N(L,t) = 0$ and $N(x,0) = u_0(x)$.

(b) Given that the solution of the new problem in (a) can be expressed as (2)

$$N(x,t) = \sum_{k=1}^{\infty} A_k e^{-D\left(\frac{k\pi}{L}\right)^2 t} \sin\left(\frac{k\pi x}{L}\right),$$

show that

$$u(x,t) = \sum_{k=1}^{\infty} A_k e^{\left[r-D\left(\frac{k\pi}{L}\right)^2\right] t} \sin\left(\frac{k\pi x}{L}\right).$$

(c) Explain the significance of this condition $L < \pi\sqrt{\dfrac{D}{r}}$ in relation to the movement of the particles in the column. (2)

<u>*Solution*</u>

Working space

Question 9 [10 marks]

Solve the following boundary value problem by setting $u(x, y) = X(x)Y(y)$,

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}, \qquad u(0, y) = 8e^{-3y}.$$

Solution

Working space

Question 10 [25 marks]

(a) Solve (20)

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 3, \quad t > o.$$

given that

$$\begin{cases} u(0,t) = u(3,t) = 0, \\ u(0,t) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x \\ |u(x,t)| < M, \qquad M \in \mathbb{R}^+. \end{cases}$$

(b) Interpret the boundary value problem. Use a diagram to illustrate your interpretation. (5)

Solution

Working space

Working space

Rough work

END OF QUESTION PAPER