UNIVERSITY OF JOHANNESBURG

APM03B3

Department of Mathematics and Applied Mathematics

Multilinear Algebra

Final Exam 2

Please read the instruction carefully:

1. Answer all questions fully.
2. Scientific calculators are allowed.
3. 100 Marks are available. 100 Marks is full Marks.

Internal Examiner: Prof E. Momoniat
External Examiner: Prof B. Jacobs

Question 1

For a simple pendulum, derive $T = 2\pi\sqrt{L/g}$, where T is the period, L the length of the pendulum and g the gravitational acceleration constant.

[10]

Question 2

For every would-be cyclist who has fought a losing battle to master shifting a user-hostile 10-speed, a Canadian company has an answer: an automatic transmission.

Instead of requiring the rider to use a shift lever to operate the derailleur that moves the chain from gear to gear, the Dugil Automatic Derailleur uses the reliable principle of centrifugal force to shift among its speeds.

Automatic bicycle shifters have appeared off and on for decades, but most were too costly or complicated to gain broad acceptance. And even though improvements in conventional derailleurs have made them easier to shift, some people are still put off by them.

With the Dugil device (the name is a contraction of the last names of the inventors, Robert Dutil and Raymond Gilbert), three weights, each mounted on two rear wheel spokes, slide outward as the speed of the wheel increases. Each weight is connected to the shift mechanism by a short rod that looks like a half spoke. As the weights move outward, they pull on the rods, which in turn actuate levers connected to a steel ring. The ring moves the derailleur from one cog to the next, much as the shifter cable does on traditional multispeed bicycles.

The rotational speed of the rear wheel determines which gear is selected, with the aim of keeping the rider pedaling at a steady cadence. As the wheel speed increases, the weights slide outward and the derailleur shifts to a higher gear. As the rider encounters hills and slows down, the weights move inward, and the derailleur shifts to a lower gear, with easier pedaling.

When the bike (and biker) is at rest, a spring mechanism automatically places the chain at the lowest gear. The Dugil unit weighs about two pounds more than a standard derailleur.

With six gears, the Dugil unit covers only the midrange of gearing available on a 10- or 12-speed bike. And unlike most multispeed bicycles, there is only a single front sprocket turned by the pedals, rather than two or three. As such, it is aimed at recreational riders rather than hard-core enthusiasts.

Bicycles equipped with the Dugil derailleur are built by Procycle Ltd., a Quebec manufacturer that makes bikes under private label for various companies. The United States importer is Autobike Inc. of South Easton, Mass. Prices range from $300 to $380.

A version of this article appears in print on Aug. 5, 1992, Section D, Page 4 of the National edition with the headline: BUSINESS TECHNOLOGY; Shifting a Bike With Centrifugal Force. Order Reprints | Today's Paper | Subscribe

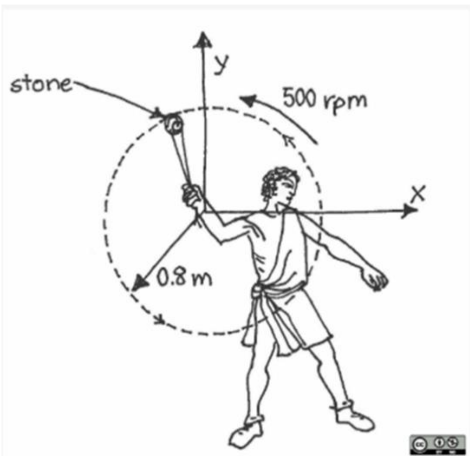
Explain how the gear system described in the article above uses centrifugal force. What are the advantages/disadvantages of this system?

[20]

Question 3

A sling holds a stone and is spun at a rate of 500 rpm in a plane perpendicular to the ground. The sling is 0.8 m long and is made from rope. At time t=0, just as the stone crosses the x axis, the rope is cut. What are the x and y coordinates of the stone 0.01 seconds later? Neglect air friction and gravity.

Show all calculations.



[10]

Question 4

Compare and contrast diffusion and heat. You may refer to Ficks' law, Fourier's law and/or the laws of thermodynamics.

[20]

Question 5

(a) Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \quad t > 0,$$

subject to the initial and boundary conditions

$$u(x,0) = x - x^2,\ u(0,t) = 0,\ u(1,t) = 0,$$

using separation of variables.

[20]

(b) Interpret the solution obtained in Question 5(a) above.

[10]

Question 6

Show how you would solve the heat equation with a source term

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x), \qquad 0 < x < L, \quad t > 0,$$

subject to the initial and boundary conditions

$$u(x,0) = f(x),\ u(0,t) = 0,\ u(L,t) = 0,$$

using separation of variables.

Do not solve the equation.

[10]

Total Marks [100]

TRIGONOMETRIC IDENTITIES

- **Reciprocal identities**

$$\sin u = \frac{1}{\csc u} \quad \cos u = \frac{1}{\sec u}$$
$$\tan u = \frac{1}{\cot u} \quad \cot u = \frac{1}{\tan u}$$
$$\csc u = \frac{1}{\sin u} \quad \sec u = \frac{1}{\cos u}$$

- **Pythagorean Identities**

$$\sin^2 u + \cos^2 u = 1$$
$$1 + \tan^2 u = \sec^2 u$$
$$1 + \cot^2 u = \csc^2 u$$

- **Quotient Identities**

$$\tan u = \frac{\sin u}{\cos u} \quad \cot u = \frac{\cos u}{\sin u}$$

- **Co-Function Identities**

$$\sin(\frac{\pi}{2} - u) = \cos u \quad \cos(\frac{\pi}{2} - u) = \sin u$$
$$\tan(\frac{\pi}{2} - u) = \cot u \quad \cot(\frac{\pi}{2} - u) = \tan u$$
$$\csc(\frac{\pi}{2} - u) = \sec u \quad \sec(\frac{\pi}{2} - u) = \csc u$$

- **Parity Identities (Even & Odd)**

$$\sin(-u) = -\sin u \quad \cos(-u) = \cos u$$
$$\tan(-u) = -\tan u \quad \cot(-u) = -\cot u$$
$$\csc(-u) = -\csc u \quad \sec(-u) = \sec u$$

- **Sum & Difference Formulas**

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

- **Double Angle Formulas**

$$\begin{aligned} \sin(2u) &= 2 \sin u \cos u \\ \cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u \\ \tan(2u) &= \frac{2 \tan u}{1 - \tan^2 u} \end{aligned}$$

- **Power-Reducing/Half Angle Formulas**

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$
$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$
$$\tan^2 u = \frac{1 - \cos(2u)}{1 + \cos(2u)}$$

- **Sum-to-Product Formulas**

$$\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$
$$\sin u - \sin v = 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$
$$\cos u + \cos v = 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$
$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

- **Product-to-Sum Formulas**

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

The sine-cosine Fourier Series

The sine-cosine Fourier series of a function $f(x)$ for which $f(x+2L) = f(x)$ is given by

$$\begin{aligned}
\frac{a_0}{2} &+ \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \\
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n > 0, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.
\end{aligned}$$

The formulae for the case when $f(x+2\pi) = f(x)$ are obtained by setting $L = \pi$.

The exponential Fourier Series

The exponential Fourier series of a function $f(x)$ for which $f(x+2L) = f(x)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad \text{where } c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx \quad n \neq 0.$$

The pure-sine Fourier Series

The pure-sine Fourier series for a function $f(x)$ defined on the region $(0, L)$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where } b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

The pure-cosine Fourier Series

The pure-cosine Fourier series for a function $f(x)$ defined on the region $(0, L)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{where } a_0 = \frac{2}{L}\int_{0}^{L} f(x)dx, \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad n > 0.$$