UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT:	**MATHEMATICS AND APPLIED MATHEMATICS**
MODULE:	**APM2B10/APM02B2** INTRODUCTION TO NUMERICAL ANALYSIS
CAMPUS:	**AUCKLAND PARK KINGSWAY**

EXAM

DATE: 25/11/2019 **SESSION: 08:30 – 10:30**

ASSESSORS **DR MV VISAYA AND MR JM HOMANN**
MODERATOR **PROF E MOMONIAT**

MARKS: 25

NUMBER OF PAGES	4 PAGES (INCLUDING COVER PAGE)
INSTRUCTIONS	INDICATE ASSIGNED NUMBER ON TOP LEFT OF SCRIPT. WRITE DOWN NUMBERS 1 TO 6 ON THE LAST PAGE . ANSWER ALL QUESTIONS AND SHOW ALL CALCULATIONS. POCKET CALCULATORS MAY BE USED. WORK TO AT LEAST THREE DECIMAL PLACES. SYMBOLS HAVE THEIR USUAL MEANING. ALL ANGLES MEASURED IN RADIANS.

Question 1 (3 marks)

Consider the function $f(x) = (x-1)\ln x$.

(a) Find the third Taylor polynomial $P_3(x)$ for f about $a = 1$.

(b) Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for the error $|R_3(0.5)|$ using the error formula, and compare it to the actual error.

Question 2 (3 marks)

Given

$$f(x) = e^{2x}\cos 3x, \quad \text{and nodes} \quad x_0 = 0, x_1 = 0.3, x_2 = 0.6,$$

construct the Lagrange interpolating polynomial for f. Do not simplify your answer.

Question 3 (3 marks)

Given the data

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3	6.8	7.1
y	102.56	113.18	130.11	142.05	167.53	195.14	224.87	256.73	299.50	326.72

set-up the equations that involve the partial of

$$S = \sum_{i=0}^{n} \big(p_n(x_i) - y_i\big)^2$$

(with respect to the unknows parameters) in constructing the least squares polynomial of degree 1. Solve the system of equations.

Question 4 (6 marks)

(a) Determine the values of n and h required to approximate

$$\int_0^2 e^{2x}\sin 3x \, \mathrm{d}x$$

accurate to within 10^{-4} with respect to the composite trapezium rule.
(Hint : $f'''(x) = 0 \implies \tan(3x) = \frac{9}{46}$)

(b) Use the composite Simpson's rule with $N = 2$ to approximate

$$\int_0^2 e^{2x}\sin 3x \, \mathrm{d}x.$$

Question 5 (7 marks)

(a) Given the differential equation

$$\frac{dy}{dx} = x - y$$

with initial value $x(1) = 1$, use the Runge-Kutta method of order two (RK2) to find A and B in the following table (next page). Explicitly show k_1 and k_2 in your solution, i.e. use

$$y_{m+1} = y_m + \frac{1}{2}(k_1 + k_2).$$

i	0	1	2	3	4
x_i	1	1.5	2	2.5	3
y_i	1	A	1.39062	B	2.15259

(b) Using information in (a), show that the characteristic function of RK2 is given by

$$F(x,y) = -\frac{3y-3x-1}{4}.$$

Use F to verify your answers to A and B in (a).

(c) Assuming that the local error in RK2 has the form

$$\epsilon_{i+1} = \frac{h^3}{6}y'''(\xi), \quad \xi \in (x_i, x_{i+1}),$$

estimate an upper bound for the global error at $x = 3$.

Question 6 (3 marks)

Use Newton's method with $\mathbf{x}^{(0)} = \mathbf{0}$ to compute $\mathbf{x}^{(1)}$ and $||\mathbf{r}^{(1)}||$ for the non-linear system

$$4x_1^2 - 20x_1 + \frac{1}{4}x_2^2 + 8 = 0,$$
$$\frac{1}{2}x_1x_2^2 + 2x_1 - 5x_2 + 8 = 0$$

where $\mathbf{x}^{(n)} = (x_n, y_n)$ and residual vector $\mathbf{r}^{(n+1)}$= $\mathbf{x}^{(n+1)}$ - $\mathbf{x}^{(n)}$.

Information

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad x_0 < \xi < x$$

$$L_i(x) = \frac{\prod_{j=0,j\neq i}^{n}(x-x_j)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)}$$

$$\Delta(x) = y(x) - p_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{2}(y_0 + y_N + 2\sum_{i=1}^{N-1} y_i)$$

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}(y_0 + y_{2N} + 4y_1 + \sum_{i=1}^{N-1}(2y_{2i} + 4y_{2i+1}))$$

$$|\Delta| \leq \left|\frac{h^2(b-a)M}{12}\right|, \quad M = \max_{[a,b]}\left|f''(x)\right|$$

$$|\Delta| \leq \left|\frac{h^4(b-a)K}{180}\right|, \quad K = \max_{[a,b]}\left|f^{(4)}(x)\right|$$

more on the next page *

$$
\begin{aligned}
k_1 &= hf(x_m, y_m) \\
k_2 &= hf(x_m + h, y_m + k_1) \\
y_{m+1} &= y_m + \frac{1}{2}(k_1 + k_2) \\
&= y_m + hF(x_m, y_m) \\
F &= \frac{1}{2}[f(x_m, y_m) + f(x_m + h, y_m + hf(x_m, y_m))]
\end{aligned}
$$

$$\alpha_m \approx 1 + hF_y(x_m, y_m)$$

$$
\begin{aligned}
y' = f(x, y) \quad &\Rightarrow \quad y'' = f_x + ff_y \\
&\Rightarrow \quad y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2
\end{aligned}
$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial x} f(x_n, y_n) & \frac{\partial}{\partial y} f(x_n, y_n) \\ \frac{\partial}{\partial x} g(x_n, y_n) & \frac{\partial}{\partial y} g(x_n, y_n) \end{bmatrix} \begin{bmatrix} x_{n+1} - x_n \\ y_{n+1} - y_n \end{bmatrix} = - \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

END OF QUESTION PAPER