APPLIED MATHEMATICS

Applied Mathematics 1A1E
APM1EB1/APM1A1E

SSA Exam: January 2020

Duration: 120 minutes | **Marks:** 50
Assessor: Mr JM Homann | **Moderator:** Dr GJ Kemp

Instructions:

1. Answer all the questions. All calculations must be shown.
2. All symbols have their usual meaning.
3. All angles are measured in degrees.
4. Only answers written in ink will be marked. Do not use red ink.
5. **Books or notes *are not* allowed.**
6. This question paper consists of 3 pages, including this one.
7. You will be penalised for using incorrect notation.
8. You will be penalised if you do not use the same notation as described in each question.
9. The use of pocket calculators is permitted. Only round off your final answer (to three decimal places - don't round off intermediate steps).
10. The test taker may answer the questions in any order, however, the test taker must clearly indicate the question number. Furthermore, rule off after each question.
11. If the test taker answers a particular question more than once, then the test taker must clearly indicate which one is to be marked by means of neatly scratching out the answers which are not to be marked. If the test taker fails to indicate which answer should be marked, then the marker will choose exactly one of the questions to mark, without complaint from the test taker.

Information:

In $\mathbb{R}^2$, O is the origin with co-ordinates $(0,0)$. In $\mathbb{R}^3$, O is the origin with co-ordinates $(0,0,0)$.

Given $P_1\left(x_1,y_1,z_1\right)$ and $P_2\left(x_2,y_2,z_2\right)$ in $\mathbb{R}^3$, then $\overrightarrow{P_1P_2}=\left(x_2-x_1,y_2-y_1,z_2-z_1\right)$ and $P_1P_2=\left|\overrightarrow{P_1P_2}\right|=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$.

Given the directed line segment $\overrightarrow{P_1P_2}$ in $\mathbb{R}^3$, recall that the direction angle α is measured from the positive x-axis to $\overrightarrow{P_1P_2}$, the direction angle β is measured from the positive y-axis to $\overrightarrow{P_1P_2}$ and the direction angle γ is measured from the positive z-axis to $\overrightarrow{P_1P_2}$. $\alpha,\beta,\gamma\in[0^\circ,180^\circ]$.

Question 1 (5 marks)

Draw $\mathbb{R}^3$ (correctly labelling the axes) and indicate, by drawing a box, $P(-2, 1, -3)$. Then draw $\overrightarrow{OP}$.

Question 2 (10 marks)

Draw a right-hand orientated three dimensional reference system and a sphere centred at the origin with a radius of $R = 1000$.

(a) Let $P(x, y, z)$ be a point with $x, y, z > 0$, let $Q(x, y, 0)$ be the projection of P onto the xy-plane, let ϕ be the angle measured from OQ to OP and let θ be the angle measured from the positive x-axis to OQ. Show that (6)

$$x = R\cos(\theta)\cos(\phi), \quad y = R\sin(\theta)\cos(\phi) \quad \text{and} \quad z = R\sin(\phi).$$

(b) Determine the x, y and z-co-ordinates of a point on the sphere with $\theta = 30°$ and $\phi = 45°$. (2)

(c) If the sphere is the Earth, determine the x, y and z-co-ordinates Brunswick, Maine, latitude $39°40'$ N and longitude $120°16'$ E, to the nearest km. Use $60' = 1°$. Latitude corresponds to ϕ and longitude corresponds to θ. (2)

Question 3 (10 marks)

The top of Mt. Dome (1605 m above sea level) is northeast of Acton (422 m above sea level). From Becton (422 m above sea level, east of Acton), the peak of Mt. Dome is seen $30°$ west of north at an angle of $22°$ above the horizontal. Taking right-handed axes with Acton as the origin, the horizontal easterly line as the z-axis, and the horizontal northerly line as x-axis:

(a) Determine the coordinates of Becton. (5)

(b) Determine the direction cosines of the of the line of sight from Acton to the top of Mt. Dome. (5)

Consider the surface of the Earth to be flat.

Question 4 (14 marks)

In $\triangle OAB$, $\overrightarrow{OA} = 6\overrightarrow{a}$ and $\overrightarrow{OB} = 6\overrightarrow{b}$. The mid- point of OA is M and the point P lies on AB such that $AP : PB = 2 : 1$. The mid-point of OP is N. Do not assume any geometric results. Only use vector algebraic and vector geometric techniques. The Auxiliary Result may be used.

(a) Calculate, in terms of $\overrightarrow{a}$ and $\overrightarrow{b}$, the vectors

i. $\overrightarrow{AB}$, (2)

ii. $\overrightarrow{OP}$, (3)

iii. $\overrightarrow{ON}$ and (2)

iv. $\overrightarrow{MN}$. (2)

(b) The line AN meets OB at C if it is continued. Given that $\overrightarrow{ON} = \overrightarrow{a} + 2\overrightarrow{b}$, determine $\overrightarrow{OC}$ in terms of $\overrightarrow{a}$ and $\overrightarrow{b}$. (5)

Question 5 (5 marks)

Given the unit vectors $\hat{a}$ and $\hat{b}$ in $\mathbb{R}^3$, determine the requirements which $\hat{a}$ and $\hat{b}$ must meet so that $\hat{a} + \hat{b}$ is also a unit vector.

HINT: Determine a requirement on the components and then the angle between $\hat{a}$ and $\hat{b}$. Make use of the scalar product.

Question 6 (6 marks)

Calculate the shortest distance between the origin and the plane which contains the points $(3, -2, -1)$, $(1, 3, 4)$ and $(2, 1, -2)$.