UNIVERSITY OF JOHANNESBURG

PROGRAM	:	BACHELOR OF ENGINEERING *MECHANICAL ENGINEERING*
SUBJECT	:	**CONTROL THEORY 4B (3B pipeline)**
CODE	:	**TKN4B21 (TKN3B21)**
DATE	:	SUMMER EXAMINATION (Supplementary) JANUARY 2020
DURATION	:	(1-PAPER) 3 hours
TOTAL MARKS	:	100

EXAMINER	:	PROF A.G.O. MUTAMBARA
EXTERNAL MODERATOR	:	PROF J. O. PEDRO (WITS)
NUMBER OF PAGES	:	8 PAGES (INCLUDING APPENDICES)

INSTRUCTIONS	:	STUDENTS CAN RETAIN QUESTION PAPERS
REQUIREMENTS	:	4 ANSWER BOOKLETS

INSTRUCTIONS TO CANDIDATES:

Notes

1. Answer **ALL** questions
2. This paper contains 5 questions
3. Number all answers according to the numbering in the question paper.
4. Make sure that you understand what the question requires before attempting it.
5. Draw proper sketches where required with all relevant information
6. **No pencil work will be marked.** Please use pen for parts to be marked.
7. Answer all questions in **English.**
8. Explain answers and give all the necessary steps – simply giving the answer is not sufficient

==

Question 1 – Dynamic system modelling and state-space representation (20)

In some mechanical positioning systems, the movement of a large object is controlled by manipulating a much smaller object that is mechanically coupled with it. The following diagram, Figure 1, depicts such a system, where a force **u(t)** is applied to a small mass m in order to position a larger mass **M**: The coupling between the objects is modeled by a spring constant **k** with a damping coefficient **b**.

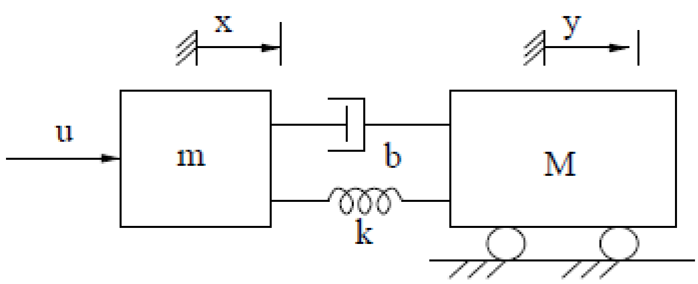


Figure 1(a)

a) Draw the free-body diagrams of the system. (3)
b) Write the equations of motion governing this system. (2)
c) Identify the appropriate state variables and express the equations of motion in the state-variable matrix form **(A, B, C, D).** (5)
d) Consider the translational mechanical system shown in Figure 1(b). Find the transfer function model:

$$\mathbf{G(s)} = \frac{X_1(s)}{F(s)} \qquad (5)$$

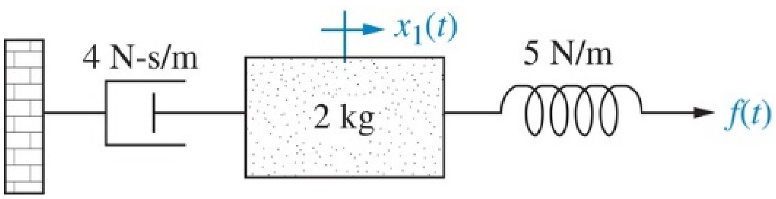


Figure 1(b)

e) For the system described by the following transfer function model derive the state variable model:

$$\mathbf{G(s)} = \frac{7s+6}{13s^3+10s^2} \tag{5}$$

Question 2 – Closed-loop system transfer function and the system response (20)

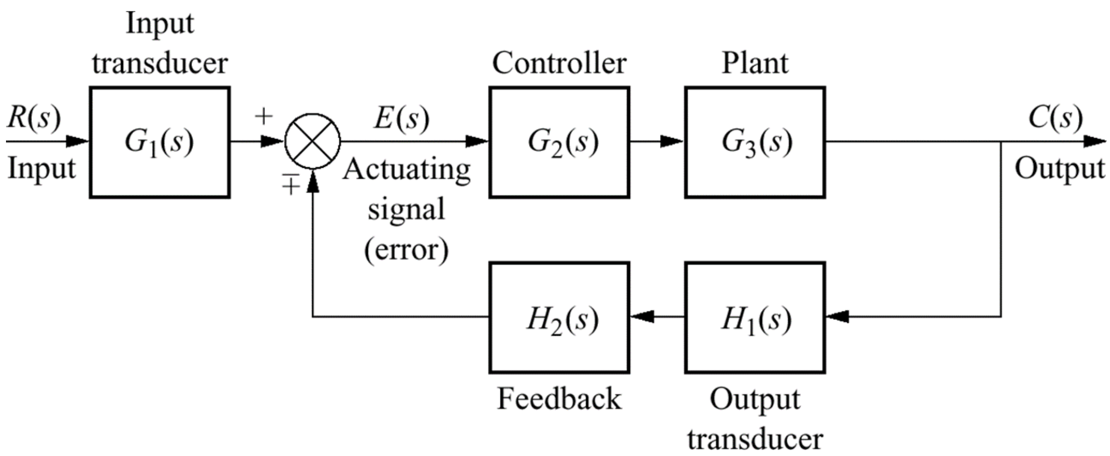


Figure 2

Consider the general (negative or positive) feedback control system illustrated in Figure 2.

a) Show that the closed-loop transfer function of the general negative feedback control system is given by

$$T(s) = \frac{C(s)}{R(s)}$$

$$= \frac{G_1(s)G_2(s)G_3(s)}{1 + G_2(s)G_3(s)H_2(s)H_1(s)} \tag{5}$$

b) For the special case of (a) where:

$G_1(s) = H_1(s) = 1;\ G_2(s) = K;\ G_3(s) = G(s)$ *and* $H_2(s) = H(s);$

Deduce a simplified expression for $T(s)$ and draw the corresponding block diagram. (2)

c) For a special case of (b) where $H(s) = 1$, deduce a further simplified expression for $T(s)$ and draw the corresponding block diagram. What name is given to such a control system? (2)

d) For the system in (c) if

$$K = 36 \ \ and \ \ G(s) = \frac{1}{s(s+6)}$$

Find ω_n, ς, T_p and T_s. (7)

e) For a more general system

$$KG(s) = \frac{K}{s(s+6)}$$

Design the controller K so that the system responds with 15% overshoot. (4)

==

Question 3 – Cascade controller (PI, PD, and PID) design **(20)**

Consider a system whose plant transfer function is given by

$$\mathbf{G(s)} = \frac{K}{(s+1)(s+3)(12)} \, .$$

In a unity feedback arrangement with a controller **D(s)** **(**i.e., a cascade compensator), a damping ratio of 0.5 and a step input;

a) Find the steady-state error for the uncompensated system. (5)
b) Show that a PI controller, **D(s)** = $1+\frac{0.1}{s}$ will drive the steady-state step response error to zero. (5)
c) Explain the impact of the PI controller on the transient response. (5)
d) From first principles show that the PID Controller has two zeros defined by

$$s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D} = 0$$

(5)

==

Question 4 – Frequency response design methods **(15)**

For a system whose plant transfer function is given by

$$\mathbf{G(s)} = \frac{2s+2}{s+10}$$

a) Draw the Bode plots (magnitude and phase angle) for the system. (10)
b) If there is a time delay of one second through the system, how does that affect the Bode plots. (5)

Question 5 – State-space representation and design via state-space **(25)**

A state-space system is represented as follows:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

Given that

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{-6} & \mathbf{-11} & \mathbf{-6} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{6} & \mathbf{2} & \mathbf{0} \end{bmatrix} \qquad \mathbf{D} = \mathbf{0};$$

a) Explain the concepts of controllability and observability (5)
b) Evaluate the controllability and observability of the system (10)
c) Show that the system's transfer function is given by

$$\mathbf{T(s)} = \frac{2}{s^2+3s+2}. \qquad (7)$$

d) Explain the significance of the pole-zero cancellations involved in the derivation of **T(s).** (3)

==

APPENDIX A: SOME CONTROL SYSTEM FORMULAE

SYSTEM TRANSFER FUNCTION

$$T(s) = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D}.$$

CONTROLLABILITY MATRIX

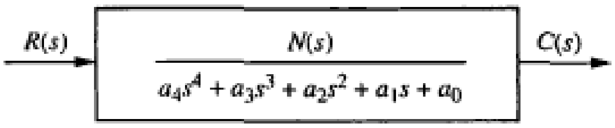
$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

OBSERVABILITY MATRIX

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

% OVERSHOOT AND ROUTH-HURWITZ TABLE

$$\boxed{\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100}$$



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

APPENDIX B

Common Laplace Transform Pairs

Time Domain Function		Laplace Domain Function
Name	Definition*	
Unit Impulse	$\delta(t)$	1
Unit Step	$\gamma(t)$ †	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Parabola	t^2	$\frac{2}{s^3}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Asymptotic Exponential	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
Dual Exponential	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
Asymptotic Dual Exponential	$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
Time multiplied Exponential	te^{-at}	$\frac{1}{(s+a)^2}$
Sine	$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
Cosine	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$
Decaying Sine	$e^{-at}\sin(\omega_d t)$	$\frac{\omega_d}{(s+a)^2+\omega_d^2}$
Decaying Cosine	$e^{-at}\cos(\omega_d t)$	$\frac{s+a}{(s+a)^2+\omega_d^2}$
Generic Oscillatory Decay	$e^{-at}\left[B\cos(\omega_d t)+\frac{C-aB}{\omega_d}\sin(\omega_d t)\right]$	$\frac{Bs+C}{(s+a)^2+\omega_d^2}$
Prototype Second Order Lowpass, underdamped	$\frac{\omega_0}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}t\right)$	$\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$
Prototype Second Order Lowpass, underdamped - Step Response	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_0^2}{s(s^2+2\zeta\omega_0 s+\omega_0^2)}$

*All time domain functions are implicitly=0 for t<0 (i.e. they are multiplied by unit step, $\gamma(t)$).
†u(t) is more commonly used for the step, but is also used for other things. $\gamma(t)$ is chosen to avoid confusion (and because in the Laplace domain it looks a little like a step function, $\Gamma(s)$).

Common Laplace Transform Properties

Name	Illustration
Definition of Transform	$f(t) \xleftrightarrow{L} F(s)$ $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$
Linearity	$Af_1(t) + Bf_2(t) \xleftrightarrow{L} AF_1(s) + BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \xleftrightarrow{L} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{L} s^2F(s) - sf(0^-) - \dot{f}(0^-)$
n^{th} Derivative	$\frac{d^n f(t)}{dt^n} \xleftrightarrow{L} s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$
Integral	$\int_0^t f(\lambda)d\lambda \xleftrightarrow{L} \frac{1}{s}F(s)$
Time Multiplication	$tf(t) \xleftrightarrow{L} -\frac{dF(s)}{ds}$
Time Delay	$f(t-a)\gamma(t-a) \xleftrightarrow{L} e^{-as}F(s)$ $\gamma(t)$ is unit step
Complex Shift	$f(t)e^{-at} \xleftrightarrow{L} F(s+a)$
Scaling	$f\left(\frac{t}{a}\right) \xleftrightarrow{L} aF(as)$
Convolution Property	$f_1(t) * f_2(t) \xleftrightarrow{L} F_1(s)F_2(s)$
Initial Value (Only if F(s) is strictly proper; order of numerator < order of denominator).	$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$
Final Value (if final value exists; e.g., decaying exponentials or constants)	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$