UNIVERSITY
OF
JOHANNESBURG

FACULTY OF SCIENCE

PHYSICS	**AUCKLAND PARK KINGSWAY CAMPUS**

PHY00A3

SUPPLEMENTARY EXAMINATION
18 JULY 2019
11:30-14:30

PHY00A3

EXAMINER: **Prof H Winkler**

EXTERNAL EXAMINER: Prof K Goldstein
Wits University

TIME: 3 HOURS **MARKS: 180**
Full Marks: 160

Please read the following instructions carefully:

ANSWER ALL QUESTIONS: 1-9

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/Question 1

QUESTION 1 **[20]**

a) Explain what the expectation value of a parameter is. Do not just provide a formula. (3)

b) A particle has the wave function $\psi(x) = \begin{cases} A\exp\left(-ibx^2\right) & a < x < a \\ 0 & |x| > a \end{cases}$

i) Normalise this wave function,
ii) Determine the expectation value of the momentum,
iii) Describe how you would verify the Heisenberg uncertainty relation for this particle (no need to do the calculations!) (14)

c) Clarify the significance of the Heisenberg uncertainty principle. (3)

QUESTION 2 **[20]**

a) Show that when the wave function ψ is an eigenfunction of the Hamiltonian, the variance of the Hamiltonian equals zero. (7)

b) Given a normalised initial wave function $\Psi(x,0) = \sqrt{\frac{30}{a^5}}x(a-x)$ in the range $0 \le x \le a$, and zero elsewhere, show that $c_n = -\frac{4\sqrt{15}}{n^3\pi^3}\left(\cos\left(n\pi\right)-1\right)$,
where c_n are the coefficients in the formula $\Psi(x,t) = \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\exp\left(-i\frac{n^2\pi^2\hbar}{2ma^2}t\right)$ (9)

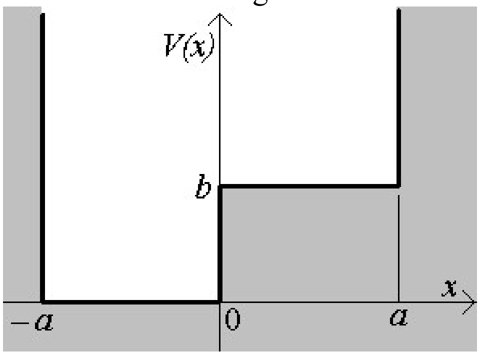
c) Show that $\hbar\omega\left(a_+a_- + \frac{1}{2}\right)\psi = E\psi$. (4)

QUESTION 3 **[20]**

a) A free particle is initially localised in the interval $-a < x < a$, and its $t = 0$ wave function $\Psi(x,0) = \sqrt{\frac{15}{16a^5}}(a^2 - x^2)$ in that range and 0 elsewhere.
i) Calculate the transform function $\phi(k)$.
ii) Hence express $\Psi(x,t)$ at a later time t. (10)

… Question 3b/

b) Consider a potential as illustrated in the diagram below.



Assume that you are asked to determine the wave function $\psi(x)$ for the bound state $0 < V(x) < b$. Draw up a set of equations representing the wave function in each region of the diagram and describe how you would go about obtaining a complete solution. (Note: do not attempt to solve these equations mathematically!) (10)

QUESTION 4 **[20]**

a) Show that if the eigenvalues of an operator are always real, then the operator is Hermitian. (4)

b) List three properties of Hermitian transformations. (3)

c) Consider the operator $\hat{Q} = \dfrac{d^2}{d\phi^2}$ where ϕ is the azimuthal angle in polar coordinates.

i) Is Q hermitian?
ii) What are the eigenfunctions and eigenvalues of Q? (8)

d) Confirm that $\left|\langle f|g\rangle\right|^2 \geq \left|\dfrac{1}{2i}\left(\langle f|g\rangle - \langle g|f\rangle\right)\right|^2$ (5)

QUESTION 5 **[20]**

a) The solution to the three-dimensional Schrödinger equation for a potential $V(r)$ contains a part dependent on the angle ϕ only, given by $\Phi(\phi) = A\exp(im\phi)$.

Prove that m must be an integer. (4)

b) Given that $P_1(x) = x$, calculate $P_1^1(x)$. (2)

c) i) Calculate the radial wave function $R_{52}(r)$ in terms of the Bohr radius a and c_0.
ii) Determine for which values of r the wave function is zero. (14)

… Question 6/

QUESTION 6 **[20]**

a) Starting with the classical definition of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, determine the quantum mechanical operators L_x, L_y and L_z. Hence show that

$$\left[L_y, L_z\right] = i\hbar L_x \ . \qquad (10)$$

b) Noting that $L_{-}f$ is an eigenfunction of both L^2 and L_z, and that $\left[L^2, L_{-}\right] = 0$, confirm that $\hat{L}^2(L_{-}f) = \lambda(L_{-}f)$. (3)

c) Consider the tensor formulation of the components of **S**. Calculate the eigenvector and eigenvalue of S_z. (7)

QUESTION 7 **[20]**

a) Why is it necessary that the combined wave function for a pair of identical particles must satisfy the relationship $\psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$? (3)

b) Consider an electron free to move in a neutral rectangular solid of sides l_x, l_y and l_z parallel to the x, y and z axes.
i) Construct a suitable Schrodinger equation for this electron.
ii) Show that the energy of this electron is given by the formula

$$E = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} + \frac{n_z^2}{l_z^2}\right)$$

where n_x, n_y and n_z are positive integers. (8)

c) Consider an electron in a crystal with atoms spaced at regular intervals parallel to the x-axis.
i) What then are the characteristics of the potential and wave function in terms of x?
ii) Without doing specific calculations, describe how the one-dimensional Schrödinger equation would be solved and the wave function determined. (8)

QUESTION 8 **[20]**

a) Two particles of mass m both move towards a common point at a common speed of 0.6 c. The particles stick together thereafter to form a new particle. Show that the mass of the combined particle is $2.5\times m$. (6)

b) Through what angle must a 0.2 MeV photon be scattered by a free electron so that it loses 10% of its energy? (7)

… Question 8c/

c) Utilising a sketch if necessary, describe the potential $V(r)$ of a diatomic molecule and explain its main features. Hence conclude that diatomic molecules will have states with energy levels approximately of the form

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad \text{where } n = 0, 1, 2, 3, \ldots \text{ (as long as } n \text{ is not too large)} \qquad (7)$$

QUESTION 9 **[20]**

a) Draw the potential experienced by an alpha particle in the vicinity of an atomic nucleus and use this to describe the alpha decay process. Explain how the alpha decay rate can be estimated (without doing any explicit calculations!) (9)

b) Determine the age of a piece of wood in which the ratio of carbon-14 to carbon-12 is 7.0×10^{-13}. (Given: The half-life of carbon-14 is 5730 years and the assumed C-14 to C-12 ratio in a living tree is 1.3×10^{-12}) (7)

c) Which of the four fundamental forces of nature only acts on hadrons? Briefly describe the characteristics of that force. (5)

END

Constants:

$c = 3 \times 10^{8}$ m.s^{-1}	$e = 1.6 \times 10^{-19}$ C	$h = 6.626 \times 10^{-34}$ J.s
$\hbar = h/2\pi$	$m_e = 9.31 \times 10^{-31}$ kg	$m_p = 1.67 \times 10^{-27}$ kg
$R = 1.097 \times 10^{7}$ m^{-1}	$a = 5.29 \times 10^{-11}$ m	$\varepsilon_0 = 8.85 \times 10^{-12}$ C^{2}m^{-2}N^{-1}
1 eV = 1.6×10^{-19} J	$R_0 = 1.2 \times 10^{-15}$ m	

Formulae:

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2} \qquad \sigma_x\sigma_p \geq \frac{\hbar}{2} \qquad \Delta E.\Delta t \geq \hbar/2$$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad p_x = -i\hbar\frac{\partial}{\partial x} \qquad f(t) = \exp\left(-\frac{iEt}{\hbar}\right) \qquad \sum_{n=1}^{\infty}|c_n|^2 = 1$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \qquad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad c_n = \sqrt{\frac{2}{a}}\int_0^a \sin\left(\frac{n\pi}{a}x\right)\Psi(x,0)$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\exp\left(-i\frac{n^2\pi^2\hbar}{2ma^2}t\right) \qquad \Psi(x,0) = \sum_{n=1}^{\infty} c_n\psi_n(x)$$

$$a_+ \equiv \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right) \qquad a_- \equiv \frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right) \quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \qquad a_-\psi_n = \sqrt{n}\psi_{n-1} \qquad \phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)\exp(-ikx)dx$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\exp\left[i\left(kx - \frac{\hbar k^2}{2m}t\right)\right]dk \qquad \Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\exp[ikx]dk$$

$$\langle f|g\rangle = \int_a^b f(x)^* g(x)dx \qquad \left|\int_a^b f(x)^* g(x)dx\right| \leq \sqrt{\int_a^b |f(x)|^2 dx\int_a^b |g(x)|^2 dx}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1)\sin^2\theta - m^2\right]\Theta = 0 \qquad \Theta(\theta) = AP_l^m(\cos\theta)$$

$$P_l^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}P_l(x) \qquad P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \qquad \int_0^{\infty}|R(r)|^2 r^2 dr = 1$$

$$\int_0^{\pi}\int_0^{2\pi}|Y(\theta,\phi)|^2\sin\theta\, d\theta\, d\phi = 1 \qquad c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)}c_j \qquad R(r) = \frac{u(r)}{r}$$

$$\rho = \kappa r = \frac{r}{an} \qquad \rho_0 = \frac{me^2}{2\pi\varepsilon_0\hbar^2\kappa} \qquad n = j_{\max} + l + 1 \qquad L_x = yp_z - zp_y$$

$$[L_x, L_y] = i\hbar L_z\ ;\ [L_y, L_z] = i\hbar L_x\ ;\ [L_z, L_x] = i\hbar L_y \qquad L_+ = L_x + iL_y\ ;\ L_- = L_x - iL_y$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} \qquad L_+ = \hbar\exp(i\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad L_- = -\hbar\exp(-i\phi)\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z f = \hbar m f \qquad L^2 f = \hbar^2 l(l+1)f \qquad [S_x, S_y] = i\hbar S_z\ ;\ [S_y, S_z] = i\hbar S_x\ ;\ [S_z, S_x] = i\hbar S_y$$

$$S^2 f = \hbar^2 s(s+1) f \qquad S_z f = \hbar m_s f \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2\right]\frac{1}{n^2} \qquad n = 1,2,3,\ldots \qquad E_F = \frac{\hbar^2}{2m}\left(3\rho\pi^2\right)^{2/3}$$

$$E_{ph} = hf = hc/\lambda \qquad \bar{x} = \gamma(x - vt) \qquad \bar{y} = y \qquad \bar{z} = z \qquad \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \bar{u}_x = \frac{u_x - v}{1 - vu_x/c^2} \qquad \bar{u}_y = \frac{u_y}{\gamma\left(1 - vu_x/c^2\right)} \qquad \bar{u}_z = \frac{u_z}{\gamma\left(1 - vu_x/c^2\right)}$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \qquad E^2 = p^2c^2 + m^2c^4 \qquad \lambda = \lambda_0 + \frac{h}{mc}(1 - \cos\theta) \qquad p = h/\lambda$$

$$E_R = \frac{\hbar^2}{I}J(J+1) \qquad E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad N = N_0 \exp(-\lambda t) \qquad T_{1/2} = \frac{0.693}{\lambda}$$

$$\int_{-\infty}^{\infty} \exp\left(-ax^2\right)dx = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{\infty} x^2 \exp\left(-ax^2\right)dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \qquad \int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2}$$