UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE	MAT8X09	FUNCTIONAL ANALYSIS A
CAMPUS	APK	
ASSESSMENT	EXAM PAPER 1	

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ASSESSOR(S)	DR A SWARTZ
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DURATION 3 HOURS	MARKS 70

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ANSWER ALL QUESTIONS.

Question 1 [8]

1. Let M be a subset of a metric space X.

   (a) What does it mean to say that M is dense in X? (1)

   (b) What does it mean to say that X is separable? (1)

   (c) What does it mean to say that M is compact? (1)

   (d) Give an example of a separable metric space. (1)

   (e) Give an example of a metric space which is not separable. (1)

   (f) Suppose (X, d) and $(Y, \tilde{d})$ are metric spaces, and that $T : X \to Y$ is continuous at x_0. Show that $x_n \to x_0 \implies Tx_n \to Tx_0$. (3)

Question 2 [10]

(a) Prove the following theorem:

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X. (5)

(b) Use the above theorem to prove that c is closed in l^{∞}. (5)

Question 3 [11]

(a) Prove the following theorem:

Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete. (7)

(b) Prove the following lemma:

A compact subset M, of a metric space is closed and bounded. (4)

Question 4 [9]

Let $T : D(T) \to Y$ be a linear operator.

(a) Prove that $N(T)$, the null space of T, is a vector space. (5)

(b) Let $D(T)$ denote the domain of T and $R(T)$ denote the range of T.

Prove that if $\dim D(T) = n < \infty$, then $\dim R(T) \leqslant n$. (4)

Question 5 [6]

Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces.

(a) If T is bounded, define what we mean by $||T||$. (1)

(b) Suppose that T is continuous at an arbitrary $x_0 \in D(T)$. Prove that T is bounded. (5)

Question 6 [8]

Let $a = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ be fixed. Let $x = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Define:

$$f : \mathbb{R}^3 \to \mathbb{R}$$

as

$$f(x) = a \cdot x = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$$

(a) Prove that f is linear. (4)

(b) Prove that $||f|| = ||a||$. (4)

Question 7 [10]

(a) Let X be a vector space. Describe what we mean by an inner product on X. Show how an inner product defines a norm. Note - you do not have to prove that the definition generates a norm. (5)

(b) Let $a, b \in \mathbb{R}$. Prove that $C[a, b]$ is not an inner product space, hence not a Hilbert space. (5)

Question 8 [8]

Prove the following lemma. For any subset $M \neq \emptyset$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$. (8)