UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS		
MODULE	**MAT 8X04**	**ABSTRACT ALGEBRA: GROUP THEORY**
CAMPUS	**APC**	
EXAM	**JULY 2019**	

DATE: JULY 2019 **SESSION: 08H00 - 12H00**

ASSESSOR(S): **RM BRITS**

EXTERNAL MODERATOR: **DR L LINDEBOOM (UNISA)**

DURATION: 4 HOURS

NUMBER OF PAGES: 4 PAGES, INCLUDING FRONT COVER
INSTRUCTIONS: ANSWER ALL THE QUESTIONS

Instructor/Course/Date: Rudi Brits/MAT8X04/

Answer all questions in the answer books provided. Answer using only black/blue ink. The exposition of your arguments/proofs should be as clear as possible. No programmable calculators allowed in the test venue.

1. Let S be a set and let G be a group of permutations of S.
   (a) Give the definition of the orbit, $\operatorname{orb}_G(i)$, of an element i of S under G. (1)
   (b) Give the definition of the stabiliser, $\operatorname{stab}_G(i)$, of $i \in S$ in G. (1)
   (c) For $\phi \in G$ give the definition of the fix, $\operatorname{fix}(\phi)$, of ϕ. (1)
   (d) Show that if G is a group of permutations on a set S, and if $s, t \in S$ belong to the same orbit, say $\operatorname{orb}_G(i)$, then $\operatorname{orb}_G(s) = \operatorname{orb}_G(t)$. (3)

2. Consider the statement of the following result, together with a short outline of its proof, and then answer the questions.

   Theorem 1 (Orbit-Stabiliser Theorem)**.** *Let G be a finite group of permutations of a set S and let $i \in S$. Then*
   $$|\operatorname{orb}_G(i)||\operatorname{stab}_G(i)| = |G|.$$

   Proof. Let $i \in S$ be arbitrary. Observe that $\operatorname{stab}_G(i)$ is a subgroup of G and hence, by Lagrange, we have that the number of cosets of $\operatorname{stab}_G(i)$ is given by $|G|/|\operatorname{stab}_G(i)|$. To obtain the result it therefore suffices to show that there is a bijective correspondence between these cosets and the set $\operatorname{orb}_G(i)$. If we let T be defined by
   $$T(\alpha \operatorname{stab}_G(i)) = \alpha(i), \quad \alpha \in G,$$
   then T is well-defined and satisfies these requirements. □

   (a) Show that $\operatorname{stab}_G(i)$ is a subgroup of G. (2)
   (b) Prove the statement in the final sentence of the outline. (4)

3. Consider the statement of Burnside's Theorem, together with a short outline of its proof, and then (referring to Question 1 and Question 2) answer the questions.

   Theorem 2 (Burnside's Theorem)**.** *If G is a finite group of permutations on a finite set S, then the number of orbits of elements of S under G is:*
   $$\frac{1}{|G|}\sum_{\phi \in G} |\operatorname{fix}(\phi)|.$$

Proof. We begin by observing that

$$\sum_{\phi \in G} |\operatorname{fix}(\phi)| = \sum_{i \in S} |\operatorname{stab}_G(i)|. \tag{1}$$

We can further deduce that if $s, t \in S$ belong to the same orbit, then

$$|\operatorname{stab}_G(s)| = |\operatorname{stab}_G(t)|. \tag{2}$$

Using this, it follows from a simple calculation that if we choose an $s \in S$ and sum over the elements of $\operatorname{orb}_G(s)$ then we obtain

$$\sum_{t \in \operatorname{orb}_G(s)} |\operatorname{stab}_G(t)| = |G|. \tag{3}$$

By summing over all the elements of S, one orbit at a time, we obtain

$$\sum_{i \in S} |\operatorname{stab}_G(i)| = |G| \times (\text{number of orbits}),$$

and hence from (1)

$$\sum_{\phi \in G} |\operatorname{fix}(\phi)| = |G| \times (\text{number of orbits}).$$

$\square$

(a) Explain why Equation (1) is true. (3)

(b) Supply the details to obtain Equation (2). (2)

(c) Do the "simple calculation" that leads to Equation (3). (4)

4. The vertices of a *non-square* rectangular tile are to be painted using 5 colours. The colour assigned to each vertex is visible on both sides of the tile. Determine the number of distinct possible designs if

   (a) The colours may be used repeatedly. (6)

   (b) No colour may be used more than once. (2)

5. (a) State, without proof, the Fundamental Theorem of Finite Abelian Groups. (2)

   (b) List, up to isomorphism, all Abelian groups of order 252. (2)

   (c) With justification, which of the groups in your list are not cyclic? (2)

   (d) With justification, which of the groups in your list are isomorphic to $\mathbb{Z}_{126} \oplus \mathbb{Z}_2$. (2)

   (e) For each of the groups in your list, write down one element of order 42 (if such an element exists). (4)

6. Suppose that G is a group of order 48. Show that the intersection of any two distinct Sylow 2-subgroups of G has order 8. (6)

7. Suppose that G is a group and $|G| = p^n m$, where p is prime and $p > m$. Prove that a Sylow p-subgroup of G must be normal in G. (4)

8. Suppose that G is a group of order 168. If G has more than one Sylow 7-subgroup, exactly how many does it have? (5)

9. Prove that a noncyclic group of order 21 must have 14 elements of order 3. (6)

10. Find up to isomorphism all groups of order 77. Motivate your arguments. (3)

Total: 65

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	6	9	8	12	6	4	5	6	3	65
Score:											