UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE	**MAT 8X04**	**ABSTRACT ALGEBRA: GROUP THEORY**
CAMPUS	**APC**	
EXAM	**JUNE 2019**	

DATE: 21 MAY 2019 **SESSION: 08H00 - 16H00**

ASSESSOR(S): **RM BRITS**

EXTERNAL MODERATOR: **DR L LINDEBOOM (UNISA)**

DURATION: 8 HOURS

NUMBER OF PAGES: 5 PAGES, INCLUDING FRONT COVER
INSTRUCTIONS: ANSWER ALL THE QUESTIONS

Answer all questions in the answer books provided. Answer using only black/blue ink. The exposition of your arguments/proofs should be as clear as possible. No programmable calculators allowed in the test venue.

1. Consider the statement of the following result, together with a brief outline of its proof.

   Theorem 1. *Let G be an Abelian group of order $p^n m$ where p is a prime that does not divide m. Then G can be written as $G = H \times K$ where*

   (i) $H = \{x \in G : x^{p^n} = e\}$

   (ii) $K = \{x \in G : x^m = e\}$

   Moreover, $|H| = p^n$.

   Proof. To show that G can be written as $G = H \times K$ we first show that H and K are normal subgroups of G. To show that $G = HK$ we take any $x \in G$, and use the fact that $\gcd(m, p^n) = 1$ to prove that x can be written as the product of an element of H and an element of K. Next, one may show that $H \cap K = \{e\}$ by proving that any $x \in H \cap K$ has order 1. By definition this establishes the fact that G can be written as $G = H \times K$. For the second assertion we observe, by using the first part of the proof, that

   $$p^n m = |HK| = |H||K|/|H \cap K| = |H||K|,$$

   and then use Cauchy's Theorem to show that $|H| = p^n$. □

   Supply the details in the above outline to show that

   (a) H and K are normal subgroups of G. (3)

   (b) $G = HK$ (3)

   (c) $H \cap K = \{e\}$ (2)

   (d) $|H| = p^n$ (3)

2. (a) Let a and b be elements of a group G. Explain what it means for a and b to be conjugate to one another. (1)

   (b) Show that conjugacy defines an equivalence relation on G. (2)

   (c) Give the definition of the conjugacy class, $\mathrm{cl}(a)$, of a. (1)

   (d) Use a bijective correspondence to show that if G is a finite group, and $a \in G$, then $|\mathrm{cl}(a)| = |G : C(a)|$ where $C(a)$ denotes the centralizer of a. (4)

(e) Let G be a non-trivial group of prime power order. Use the class equation to show that G has a non-trivial centre. (5)

(f) It is well-known that if G is a group, and $G/Z(G)$ is cyclic then G is Abelian. Use this fact to show that if the order of a group G is the square of a prime then G is Abelian. (3)

3. Consider the statement of the following result, together with a brief outline of its proof, and then answer the questions.

Theorem 2 (First Sylow Theorem)**.** *Let G be a finite group and let p be a prime. If p^k divides $|G|$, then G has at least one subgroup of order p^k.*

Proof. The proof is by induction on $|G|$. If $|G| = 1$ then the statement of the theorem holds trivially. Now assume the statement of the theorem is true for all groups of order less than $|G|$. If G has a proper subgroup H such that p^k divides $|H|$ then H has a subgroup of order p^k and we are done. Thus we may assume that p^k does not divide the order of any proper subgroup of G. Next use the class equation for G to show that p divides $|Z(G)|$, and then Cauchy's Theorem for Abelian Groups to infer the existence of an x in the centre of G with $|x| = p$. Form now the factor group $G/\langle x\rangle$ and deduce that $G/\langle x\rangle$ contains a subgroup of order p^{k-1}. We can then show that this subgroup has the form $H/\langle x\rangle$ where H is a subgroup of G. Finally, to complete the proof, note that $|H/\langle x\rangle| = p^{k-1}$ and $|\langle x\rangle| = p$ together implies that $|H| = p^k$. But this leads to a contradiction from which the theorem then follows. □

(a) In lines 3-4 of the outline, why are we done if H has a subgroup of order p^k? (1)

(b) In lines 5-6 of the outline, supply the details involving the class equation to show that p divides $|Z(G)|$. (4)

(c) In line 8 of the outline, explain why $G/\langle x\rangle$ contains a subgroup of order p^{k-1}. (1)

(d) In line 8-9 of the outline, prove the statement that the subgroup has the form $H/\langle x\rangle$ where H is a subgroup of G. (5)

(e) In the last sentence of the outline explain the contradiction, and why the theorem then follows. (2)

4. Consider the statement of the following result, together with a brief outline of its proof, and then answer the questions.

Theorem 3 (Second Sylow Theorem)**.** *If H is a subgroup of a finite group G and $|H|$ is a power of a prime p, then H is contained in some Sylow p-subgroup.*

Proof. Let K be a Sylow p-subgroup of G and let

$$C = \{K_1, K_2, \ldots, K_n\}$$

(with $K = K_1$) be the set of all conjugates of K in G. Since conjugation is an isomorphism, each element of C is a Sylow p-subgroup of G. Now let S_C denote the group of all permutations of C. For each $g \in G$ we define

$$\phi_g : C \to C \text{ by } \phi_g(K_i) = gK_ig^{-1}.$$

It is now easy to show that each $\phi_g \in S_C$. Next define a mapping

$$T : G \to S_c \text{ by } T(g) = \phi_g.$$

Then it is easy to show that T is a homomorphism from G to S_C. Consider now the image $T(H)$ of H under the homomorphism T. It follows that, for each i, $\left|\mathrm{orb}_{T(H)}(K_i)\right|$ is a power of p, and further that

$$\left|\mathrm{orb}_{T(H)}(K_i)\right| = 1 \Leftrightarrow H \leq K_i.$$

So if we look at the conclusion of the theorem, to complete the proof, all we need to do is to show that, for some i, $\left|\mathrm{orb}_{T(H)}(K_i)\right| = 1$.
To do this we start by first proving that $|C| = |G : N(K)|$ and then use this to show that $|C|$ is not divisible by p. Because the orbits, $\mathrm{orb}_{T(H)}(K_i)$, partition C and each $|\,\mathrm{orb}_{T(H)}(K_i)|$ is a power of p it follows that $|C|$ is the sum of powers of p. If no orbit has size 1 (i.e p^0) then, for each i, $|\,\mathrm{orb}_{T(H)}(K_i)|$ is divisible by p and so $|C|$ would be divisible by p which contradicts the preceding statement. Thus for some i we must have $|\,\mathrm{orb}_{T(H)}(K_i)| = 1$ and the result follows. □

(a) Give the definition of a Sylow p-subgroup. (2)

(b) In lines 3-4 show that conjugation is an isomorphism. (2)

(c) In line 7 show that each $\phi_g \in S_C$. (2)

(d) In line 9 show that T is a homomorphism from G to S_C. (2)

(e) In lines 10-12 show that $\left|\mathrm{orb}_{T(H)}(K_i)\right|$ is a power of p, and that $\left|\mathrm{orb}_{T(H)}(K_i)\right| = 1 \Leftrightarrow H \leq K_i$. (6)

(f) In lines 15-16 show that $|C| = |G : N(K)|$ and consequently that that $|C|$ is not divisible by p. (6)

5. Let G be a group of order n where n is not a prime number, and let p be a prime divisor of n. If 1 is the only divisor of n that is equal to 1 modulo p, then show that G is not a simple group. (5)

Total: 65

Question:	1	2	3	4	5	Total
Points:	11	16	13	20	5	65
Score:						

Instructor/Course/Date: Rudi Brits/MAT8X04/

Answer all questions in the answer books provided. Answer using only black/blue ink. The exposition of your arguments/proofs should be as clear as possible. No programmable calculators allowed in the test venue.

1. The five spokes of a round wheel, arranged at 72 degree intervals around the axis of the wheel, are to be painted using 6 colours. The colour assigned to each spoke is visible on both sides of the wheel. Use Burnside's Theorem to determine the number of distinct possible designs if the colours may be used repeatedly. (6)

2. (a) Let G be an Abelian group of order 100. Find and write down all the possible isomorphism classes for G. (2)

   (b) Let G be an Abelian group of order 100. Show that if G has exactly one subgroup for each divisor of 100 that G is cyclic. (4)

   (c) Let G be any finite Abelian group. Show that if G has exactly one subgroup for each divisor of $|G|$ that G is cyclic. (5)

3. Let G be a group of order 105.

   (a) Determine the possibilities for the number of Sylow 5-subgroups of G and the possibilities for the number of Sylow 7-subgroups of G. (4)

   (b) Show that G has either one Sylow 5-subgroup or it has one Sylow 7-subgroup. (4)

   (c) Show that G has a subgroup of order 35. (3)

4. Let G be a group with $|G| = pqr$ where p, q and r are prime numbers.

   (a) Show that if $p = q = r$ then G is not a simple group. (2)

   (b) Use the Index Theorem to show that if $p = q \neq r$ then G is not a simple group. (4)

   (c) Use Sylow's Third Theorem to show that if $r < q < p$ then G is not a simple group. (6)

   (d) What conclusion can be made for groups of order pqr where p, q and r are prime numbers? (1)

Total: 40

Question:	1	2	3	4	Total
Points:	6	11	11	13	41
Score:					