University of Johannesburg

Faculty of Science

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS

MODULE **MAT3A20/MAT02A3**
DISCRETE MATHEMATICS

CAMPUS **APK**

EXAM **JUNE 2019**

Date 08/06/2019	Session 08:30 – 11:00
Examiner	Dr W Morton
External Examiner	Dr R Kellerman
Duration 150 Minutes	70 Marks

Surname and initials: ______________________

Student number: ______________________

Tel No.: ______________________

INSTRUCTIONS:

1. The paper consists of **12** printed pages, **excluding** the front page.
2. Read the questions carefully and answer all questions.
3. **Write out all calculations (steps) and motivate all answers.**
4. Questions are to be answered on the question paper in the space provided. Please indicate when the blank side of a page is used.
5. Non-programmable calculators are allowed.

Question 1 [4]

Let $A = \{\emptyset, \{1, 2\}\}$ and $B = \{1, 2\}$. Complete the following:

(a) $A \cap B =$____ (1)

(b) $\mathscr{P}(A) =$____ (1)

(c) $A \times B =$____ (1)

(d) $A - (A - B) =$____ (1)

Question 2 [4]

Let S be the set of all finite strings of a's and b's (i.e., $S = \{a, b, aa, ab, ba, bb, aaa, ...\}$) and define $d : S \to \mathbb{Z}$ as follows, for $s \in S$,

$$d(s) = \text{ number of } a\text{'s minus the number of } b\text{'s in } s.$$

(a) Is d injective? Explain. (1)

(b) Is d surjective? Explain. (2)

(c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that $fd : S \to \mathbb{Z}$ is surjective. Is f surjective? Explain. (1)

Question 3 [4]

Let $\equiv_5 \subseteq \mathbb{Z} \times \mathbb{Z}$ be the relation of having the same remainder after division with 5.

(a) Is $[18]_{\equiv_5} = [23]_{\equiv_5}$? Explain. (1)

(b) Complete: $\mathbb{Z}/_{\equiv_5} =$ ______________________________ (1)

(c) Show that $\equiv_5$ is transitive. (2)

Question 4 [4]

Let A, B and C be sets. Prove that $(A - B) - C = A - (B \cup C)$.

Question 5 [4]

(a) Complete the truth-table: (2)

p	q	r	$\neg(p \leftrightarrow \neg r)$	$\neg p \vee (q \wedge r)$	$r \rightarrow q$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

(b) A set $\mathcal{S}$ of propositional formulas is said to be **independent** if, and only if, for every formula $F \in \mathcal{S}$, F is not a logical consequence of $\mathcal{S} - \{F\}$. Is $\{\neg(p \leftrightarrow \neg r), \neg p \vee (q \wedge r), r \rightarrow q\}$ independent? Motivate. (2)

Question 6 [2]

Consider the following conditional statement: There are as many rational numbers as there are irrational numbers only if the set of all irrational numbers is infinite.

(a) Rephrase the statement in the form "__ is necessary for __". (1)

(b) State the converse of the implication. (1)

Question 7 [4]

Determine whether or not the following formula is a tautology using a semantic tableaux.

$$((p \wedge q) \to r) \to ((p \to r) \vee (q \to r)).$$

Question 8 [3]

Make use of known equivalences to construct a formula in conjunctive normal form (CNF) equivalent to the following formula:

$$((p \vee q) \to \neg r) \to \neg(\neg q \wedge r).$$

Question 9 [4]

(a) How many nonnegative integer solutions does $x_1 + x_2 + x_3 = 12$ have? (1)

(b) How many integer solutions does $x_1 + x_2 + x_3 = 12$ have, if $0 \leq x_i \leq 5$ for $i = 1, 2, 3$? (3)

Question 10 [2]

Determine the coefficient of $x^6y^6z^2$ in the expansion of $(2x^2 + 3y^3 - z)^7$.

Question 11 [4]

Let $X = \{a, b, c, d, e, f, g, h\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a) How many injective functions are there from X to Y? Explain. (2)

(b) How many injective functions are there from Y to X? Explain. (2)

Question 12 [6]

Consider the recurrence relation $a_n = 4a_{n-1} + (-1)^n$ for $n > 0$ with initial conditions $a_0 = 1$. Solve the recurrence relation using generating functions.

Question 13 [7]

(a) Draw the complete graph K_5 of order 5. (1)

(b) What is K_5's size? (1)

(c) $\chi(K_5) =$ ____________________ (1)

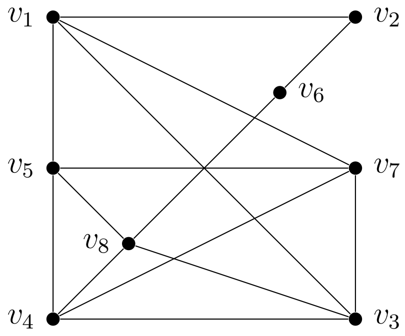
(d) Prove that K_5 is nonplanar. (2)

(e) Formulate the 5-colour theorem. (1)

(f) Hence, does the converse of the 5-colour theorem hold? Explain. (1)

Question 14 [10]

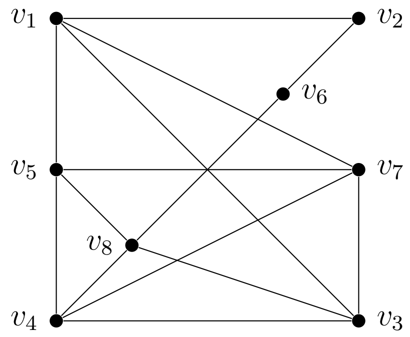
Consider the graph G depicted below and answer the questions that follow:



(a) Is G Eulerian? If so, find an Eulerian circuit. If not, explain why not. (3)

(b) Determine whether or not the graph G is planar. If yes, draw it as a plane graph. If not, use Kuratwoski's theorem to prove that it is not. (3)

Recall that G is the graph depicted below:

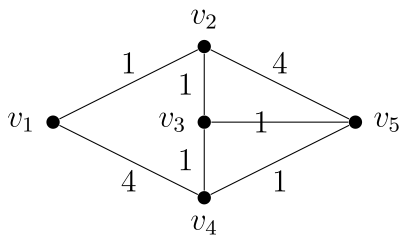


(c) Is G bipartite? If so, give the two partite sets; if not, explain why not. (2)

(d) The **girth** of a graph is the length of its shortest cycle. Find the girth of G. (2)

Question 15 [3]

Use Dijkstra's algorithm to calculate the shortest path from v_1 to v_5 in the following graph:



Question 16 [5]

Prove that every connected graph has a spanning tree.