FACULTY OF SCIENCE

UNIVERSITY OF JOHANNESBURG

DEPARTMENT OF PURE AND APPLIED MATHEMATICS	
MODULE	**MAT01A2 / MAT2A10** **Sequences, Series and Vector Calculus**
CAMPUS	**APK**
ASSESSMENT	**SUPPLEMENTARY EXAMINATION**

DATE	**JULY 2019**
ASSESSOR(S)	**M. SIAS**
INTERNAL MODERATOR	**C. MARAIS**
DURATION: 120 MINUTES	**MARKS: 50**

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

CONTACT NUMBER ______________________________

NUMBER OF PAGES: 1 + 13 PAGES

INSTRUCTIONS:

1. ANSWER ALL THE QUESTIONS ON THE PAPER IN PEN.
2. NO CALCULATORS ARE ALLOWED.
3. SHOW ALL CALCULATIONS AND MOTIVATE ALL ANSWERS.
4. IF YOU REQUIRE EXTRA SPACE, CONTINUE ON THE FACING BLANK PAGE AND INDICATE THIS CLEARLY.

Question 1

For questions (1.1) - (1.5), please circle only **ONE** correct answer: [5]

(1.1) Given the series:

$$A := \sum_{m=1}^{\infty} \frac{(-1)^m}{m^{0.1}} \text{ and } B := \sum_{m=1}^{\infty} (-1)^m \frac{2^m}{m^{100}}.$$

Determine whether each series is convergent or divergent. (1)

(a) A is convergent, B is divergent.
(b) A is divergent, B is convergent.
(c) The series are both convergent.
(d) The series are both divergent.

(1.2) If a series $\sum a_n$ is conditionally convergent, then $\lim_{n\to\infty} |a_n| \neq 0$. (1)

(a) True (b) False

(1.3) The Root Test can be used to determine whether the series $\sum e^n \left(1 + \frac{1}{n}\right)^{-n^2}$ converges. (1)

(a) True (b) False

(1.4) Suppose that the series $\sum_{n=0}^{\infty} c_n (x-2)^n$ converges when $x = 4$ and diverges when $x = -4$. What can be said about the convergence or divergence of the following series: (1)

$$C := \sum_{n=0}^{\infty} c_n (-1)^n \text{ and } D := \sum_{n=0}^{\infty} c_n 7^n.$$

(a) C is convergent, D is divergent.
(b) C is divergent, D is convergent.
(c) The series are both convergent.
(d) The series are both divergent.

(1.5) A power series representation and radius of convergence for $f(x) = \frac{1}{4+x^2}$ is: (1)

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$; $R = 1$.
(b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$; $R = 1$.
(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^{n+1}}$; $R = 2$.
(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$; $R = 2$.

Question 2 [4]

Show that the $\lim_{n\to\infty} \dfrac{3n^2+1}{4n^2+1} = \dfrac{3}{4}$ by using the precise definition of a limit of a sequence.

Question 3 [2]

Give an example of two divergent sequences $\{a_n\}$ and $\{b_n\}$ such that the sequence $\{a_n b_n\}$ is convergent.

Question 4 [4]

Let $a_1 = \sqrt{3}$ and let $a_{n+1} = \sqrt{3a_n}$ for $n \in \mathbb{N}$. Show that $\{a_n\}$ is increasing and bounded above by 3, and find its limit.

Question 5

Test the following series for convergence or divergence: [5]

(5.1) $\sum_{n=1}^{\infty}(-1)^n\frac{(n!)^n}{n^{3n}}$ (3)

(5.2) $\sum_{n=1}^{\infty}\frac{\cos 2n}{1+3^n}$ (2)

Question 6

State and prove the Alternating Series Test. [4]

Question 7

Find a Maclaurin series for the given function: [3]

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0. \end{cases}$$

Question 8

Use the binomial series series to expand f and state its radius of convergence: [4]

$$f(x) = (1-x)^{\frac{2}{3}}\,.$$

Question 9

Determine $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \left\langle \frac{1}{1+t^2}, \cos^2 t, te^{t^2} \right\rangle$ and $\mathbf{r}(0) = \langle 1, 0, 1 \rangle$. [3]

Question 10

Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t: [4]

$$\mathbf{r}(t) = \langle 2\sin t, 4, 2\cos t\rangle .$$

Question 11

State the definition of the curvature of a smooth curve C. [2]

Question 12

Show that if there is a $c \in \mathbb{R}$ such that $|\mathbf{r}(t)| = c$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$. [2]

Question 13

Prove that the curvature of a curve C with vector function $\mathbf{r}(t)$ is given by the following formula:[4]

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Question 14

A particle moves with position function

$$\mathbf{r}(t) = \left\langle t^3, 1 - t^2, t + 7 \right\rangle .$$

Determine the normal component of the accelaration of the particle. [4]