FACULTY OF SCIENCE

UNIVERSITY OF JOHANNESBURG

DEPARTMENT OF PURE AND APPLIED MATHEMATICS

MODULE	**MAT01A2 / MAT2A10** **Sequences, Series and Vector Calculus**
CAMPUS	**APK**
ASSESSMENT	**EXAMINATION**

DATE: 06/06/2019	**TIME: 08:30**
ASSESSOR(S)	**M. SIAS**
INTERNAL MODERATOR	**C. MARAIS**
DURATION: 120 MINUTES	**MARKS: 50**

SURNAME AND INITIALS ______________________________

STUDENT NUMBER ______________________________

CONTACT NUMBER ______________________________

NUMBER OF PAGES: 1 + 11 PAGES

INSTRUCTIONS:

1. ANSWER ALL THE QUESTIONS ON THE PAPER IN PEN.
2. CALCULATORS ARE ALLOWED.
3. SHOW ALL CALCULATIONS AND MOTIVATE ALL ANSWERS.
4. IF YOU REQUIRE EXTRA SPACE, CONTINUE ON THE <u>FACING</u> BLANK PAGE AND INDICATE THIS CLEARLY.

Question 1

For questions (1.1) - (1.5), please circle only **ONE** correct answer: [5]

(1.1) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=1}^{\infty} \frac{(-1)^k \arctan k}{k^5}.$$

Select the correct answer. (1)

(a) divergent

(b) absolutely convergent

(c) conditionally convergent

(1.2) Determine the sum of the series: $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \ldots$ (1)

(a) $1 - e^3$ (b) $e^3 - 1$ (c) e^3 (d) $\frac{e^3}{2}$ (e) $\frac{e^3}{3}$.

(1.3) Find the Maclaurin series expansion of $x\cos(4x)$. (1)

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n+1}}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n+1}}{(2n)!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 4^{2n} x^{2n+1}}{(2n!)}$

(1.4) Let C be a smooth curve defined by a vector function $\mathbf{r}$ with tangent vector $\mathbf{T}$, binormal vector $\mathbf{B}$ and normal vector $\mathbf{N}$. Then the following statements are true : (1)

(i) $\mathbf{T} \perp \mathbf{T}'$ (ii) $\mathbf{T} \perp \mathbf{B}$ (iii) $\mathbf{B} \perp \mathbf{N}$

(a) i, ii, & iii (b) only i & ii (c) only ii & iii (d) only i & iii

(1.5) Find $\lim_{t\to\infty} \overline{\mathbf{r}}(t)$, where $\overline{\mathbf{r}}(t) = \langle \arctan t, e^{-7t}, \frac{\ln t}{t} \rangle$. (1)

(a) $\langle \pi, 0, 0 \rangle$.

(b) $\langle \frac{\pi}{2}, 0, 0 \rangle$.

(c) $\langle 0, 0, 0 \rangle$.

(d) $\langle 1, 1, 0 \rangle$.

Question 2

Show that $\lim_{n\to\infty} \dfrac{n^2+n}{2n^2+1} = \dfrac{1}{2}$ by using the precise definition of a limit of a sequence. [4]

Question 3 [5]

(3.1) Suppose that $\sum a_n$ and $\sum b_n$ are infinite series with positive terms and that $\sum b_n$ is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty,$$

then $\sum a_n$ is also divergent. (2)

(3.2) Use part (3.1), or otherwise, to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent. (3)

Question 4

Prove the following: If $\lim_{n\to\infty} |a_n|^{1/n} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. [4]

Question 5

Prove that

$$\ln(5-x) = \ln 5 - \frac{1}{5}\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^n(n+1)}$$

and determine the radius of convergence for this series. [5]

Question 6 [5]

(6.1) Use the binomial series to expand (4)

$$f(x) = \sqrt[4]{1 + x^6}.$$

(6.2) Find the radius of convergence of the series in (6.1). (1)

Question 7

Use power series to evaluate [4]

$$\lim_{x \to \infty} \frac{\cos(x^2) - 1}{x^4}$$

Question 8

Let $\overline{\mathbf{r}}(t)$ be a vector given by [3]

$$\overline{\mathbf{r}}(t) = \langle \tan\frac{1}{t}, \frac{2t-1}{3t+1}, te^{-2t} \rangle.$$

(8.1) Find the domain of $\overline{\mathbf{r}}(t)$. (2)

(8.2) Evaluate $\lim_{t\to\infty} \overline{\mathbf{r}}(t)$. (1)

Question 9

Let C be a smooth curve. [8]

(9.1) Prove that the curvature (4)

$$\kappa(t) = \frac{|\overline{\mathbf{r}}'(t) \times \overline{\mathbf{r}}''(t)|}{|\overline{\mathbf{r}}'(t)|^3}.$$

(9.2) Find the curvature of $\overline{\mathbf{r}}(t) = \langle \sqrt{15}t, e^t, \sin(t) \rangle$ at $(0, 1, 0)$. (4)

Question 10

Find the velocity, acceleration and speed of a particle with the position function: [3]

$$\bar{\mathbf{r}}(t) = \cos(-2t)\hat{\mathbf{i}} + \sin(-2t)\hat{\mathbf{j}}, \quad \text{at } t = 0.$$

Describe the path of the particle.

Question 11

At what point do the curves $\overline{\mathbf{r}}(t) = \langle t,\, 1-t,\, 3+t^2\rangle$ and $\overline{\mathbf{u}}(s) = \langle 3-s,\, s-2,\, s^2\rangle$ intersect? Find their angle of intersection. [4]