APPLIED MATHEMATICS

Quantum Computing
APM8X16

Supplementary Examination: June/2019

Duration: 120 minutes **Marks:** 50
Assessor: Dr. G.J. Kemp
Moderator: Prof. Y. Hardy

Surname: ____________________ **Student Number:** ______________

Question 1 (15 marks)
Consider a qubit state preparation that has produced the ensemble $\{p_i, |\psi_i\rangle\}$, where each $|\psi_i\rangle$ is a pure state and p_i is the corresponding probability.

(a) Show that for each pure state $|\psi_i\rangle$, (6)

$$|\psi_i\rangle\langle\psi_i| = \frac{1}{2}\left(I + \boldsymbol{r}_i \cdot \boldsymbol{\sigma}\right),$$

where $\boldsymbol{r}_i$ is a three-dimensional unit vector indicating the position of the state on the Bloch sphere.

(b) For an arbitrary mixed state, show that the density matrix can be written as (5)

$$\rho = \frac{1}{2}\left(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}\right),$$

where $|\boldsymbol{r}| < 1$.

(c) Where would the completely mixed state $\rho = I/2$ be located on (or inside) the Bloch sphere? (2)

(d) Where would the pure state $|+\rangle$ and $|-\rangle$ be located on the Bloch sphere? (2)

Question 2 (12 marks)
This question discusses the idea of ensemble ambiguity.

(a) Consider two ensembles $\{p_i, |\psi_i\rangle\}$ and $\{q_i, |\phi_i\rangle\}$. If the two sets of states are related to each other by (6)

$$\sqrt{p_i}\,|\psi_i\rangle = \sum_j u_{ij}\sqrt{q_j}\,|\phi_j\rangle,$$

where u_{ij} is a unitary matrix, show that both ensembles give the same density matrix.

(b) Study the following density matrix (6)

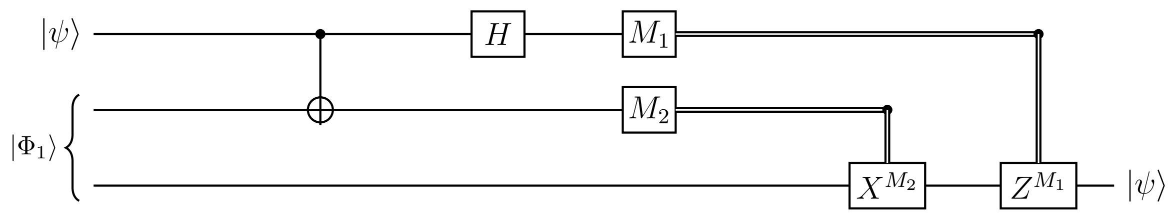
$$\rho = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}.$$

Solving the eigenvalue problem for ρ one may conclude that the system described by ρ has the ensemble

$$\rho = \frac{3}{4}|+\rangle\langle+| + \frac{1}{4}|-\rangle\langle-|.$$

Find one other ensemble that gives the same density matrix ρ.

Question 3 (14 marks)
The following circuit implements the quantum teleportation algorithm.



The top two registers belong to Alice and the bottom one belongs to Bob. Alice and Bob share the entangled state $|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice wants to transit the qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob using the above circuit.

(a) Calculate the overall state just before Alice performs the measurements M_1 and M_2. (6)

(b) What is the density matrix for the system after Alice has performed the measurements M_1 and M_2? (4)

(c) Say the outcomes for measurements are $M_1 = 1$ and $M_2 = 0$. Alice must now send these classical bits of information to Bob so that he may recover the original state $|\psi\rangle$. What transformations must Bob do to his qubit to recover $|\psi\rangle$ (also state the correct order that the transformations need to be applied in)? (4)

Question 4 (9 marks)

Find the Schmidt decomposition for the states

(a) (2)

$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

(b) (2)

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

(c) Find the Schmidt coefficients for the following state (5)

$$\frac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}}$$

Useful information

- The Pauli matrices are defined as:

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The Hadamard transform H is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- We also define

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

- The Schmidt decomposition for a pure state $|\psi\rangle$ of a composite system AB is

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle,$$

where the λ_i are the Schmidt coefficients and the $|i_{A/B}\rangle$ are orthonormal states for systems A and B respectively.

END OF QUESTION PAPER