UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

Applied Mathematics Honours	**APK Campus**
APM8X03 Relativity A	
JULY 2019	

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TIME 3 Hours **MARKS** 100

Please read the following instructions carefully

1. Answer all the questions.
2. Answer each of the three questions in a separate booklet.
3. You may use a calculator.
4. This paper consists of 4 pages, including this one.
5. Some useful results are listed on the last page of this paper and may be used without proof.

COURSE:	APPLIED MATHEMATICS HONOURS	**TIME:**	3 HOURS
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QUESTION 1

(a) Elliptic cylindrical coordinates are defined by

$$x = a\cosh u \cos v; \quad y = a \sinh u \sin v; \quad z = z,$$

where

$$u \geq 0; \quad 0 \leq v < 2\pi; \quad -\infty < z < \infty \ .$$

(i) Sketch the u- and v-curves in the xy plane.
(HINT: $\cos^2 v + \sin^2 v = 1$ and $\cosh^2 u - \sinh^2 u = 1$.) (7)

(ii) Obtain the tangential base $\{\mathbf{e}_i\}$ for (u, v, z). (4)

(iii) Derive the metric tensor for (u, v, z). (5)

(b) Show that $\mathbf{T}$ is a tensor of rank 2 if and only if $K = T^i_j X^j_i$ is invariant for all tensors $\mathbf{X}$. (8)

(c) T^k are the components of a contravariant tensor of rank 1 and g_{ij} are the components of a covariant tensor of rank 2. Show that $g_{ij}T^j$ are the components of a covariant tensor of rank 1. (4)

(d) $\mathbf{T}$ is a tensor of rank 2 in $\mathbb{R}^N$. Show that $T_{(ij)} - \dfrac{1}{N} T^k_k \, g_{ij}$ is traceless, where the notation $T_{(ij)}$ denotes a symmetrised tensor. (7)

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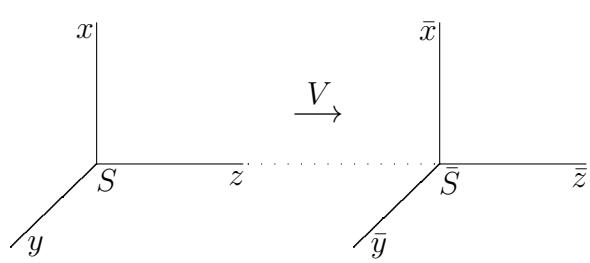
QUESTION 2

In the figure two inertial systems S and $\bar{S}$ are shown. $\bar{S}$ moves at a constant speed V with respect to S. The coordinate systems are chosen such that the relative direction of motion is the z-direction in both frames.



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(a) Define the coordinates x^μ, $\mu = 1, 2, 3, 4$ in Minkowski space, denoting the invariant speed of light by c. Assume that the coordinate transformation $x^\mu \to \bar{x}^\nu$ is continuous and show how the transformation can be expanded as a series.(4)

(b) Show how the series in (a) is simplified if a time synchronisation between S and $\bar{S}$ is assumed. (2)

(c) According to Einstein's postulates the form of the physical laws are the same in all inertial frames. Show that this assumption reduces the transformation in (b) to a linear form, i.e. $\bar{x}^\mu = a^\mu{}_\nu\, x^\nu$. (4)

(d) By the use of symmetry arguments, the matrix of coefficients in (c) can be simplified to assume the form

$$(a^\mu{}_\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a^3{}_3 & a^3{}_4 \\ 0 & 0 & a^4{}_3 & a^4{}_4 \end{pmatrix}.$$

Consider the origin of $\bar{S}$ as an event in order to find a relation between $a^3{}_3$ and $a^3{}_4$, introducing the definition $\beta := \frac{V}{c}$ in the process. (5)

(e) The light-sphere thought experiment yields the relations $\eta_{\mu\nu}\, a^\mu{}_\alpha\, a^\nu{}_\beta = \eta_{\alpha\beta}$ between the coefficients in the transformation, where the Minkowski metric is given by

$$(\eta_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Make use of these relations as well as the result of (d) in order to complete the Lorentz transformation, in terms of β and $\gamma := \frac{1}{\sqrt{1-\beta^2}}$. (12)

(f) The Lorentz transformation $S \to \bar{S}$ as derived above is given by $\bar{\mathbf{x}} = L\mathbf{x}$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix}, \quad \beta = \frac{V}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Invert this transformation by inverting the Lorentz matrix L and summarise the effect of the inversion. (8)

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QUESTION 3

(a) Use the Minkowski metric as defined in 2.(e) to write the line element ds in Minkowski space as a multiple of the time element dt by introducing the quantities $\beta_v = \dfrac{v}{c}, \quad \gamma_v = \dfrac{1}{\sqrt{1-\beta_v^2}}$. (3)

(b) The velocity components of a moving particle in the S system in Question 2 is given by $v^k = \dfrac{dx^k}{dt}$ and in the $\bar{S}$ system by $\bar{v}^k = \dfrac{d\bar{x}^k}{d\bar{t}}$. Define a four-velocity in Minkowski space and make use of it in order to derive the velocity transformation

$$\bar{v}^{1,2} = \frac{Q}{\gamma} v^{1,2}, \quad \bar{v}^3 = Q(v^3 - V), \quad Q = \frac{1}{1 - \frac{Vv^3}{c^2}}. \qquad (13)$$

(c) An electron e^- has a kinetic energy of $\frac{19}{20} m_e c^2$, where m_e is the *rest mass* of the electron and c is the speed of light. It makes a head-on collision with a positron e^+ that is at rest. (A positron has the same mass as an electron but opposite charge.) In the collision the two particles annihilate each other and produce two photons γ of equal energy. The reaction can be written as

$$e^- + e^+ \to 2\gamma.$$

Answer the following questions, giving all results in terms of m_e and c.

(i) Determine the speed at which the electron was travelling before it collided with the positron. (6)

(ii) What is the momentum of the electron? (2)

(iii) What is the total energy of the electron? (2)

(iv) What is the total energy of the positron? (3)

(v) What is the energy of each photon? (3)

(vi) What is the momentum of each photon? (3)

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Useful Results

Lorentz transformation

$$\beta := \frac{V}{c},\ \gamma := \frac{1}{\sqrt{1-\beta^2}}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \gamma(x^3 - \beta\, x^4),\ \bar{x}^4 = \gamma(-\beta\, x^3 + x^4)$$

$$\bar{x} = x,\ \bar{y} = y,\quad \bar{z} = \gamma(z - \beta\, ct),\quad \bar{t} = \gamma\left(t - \tfrac{\beta}{c}\, z\right)$$

Relativistic Dynamics

$$\beta_v := \frac{v}{c},\ \gamma_v := \frac{1}{\sqrt{1-\beta_v^2}}$$

$$m = \gamma_v m_0$$

$$E \ = \ mc^2 \ = \ T + m_0 c^2$$

$$E^2 = p^2c^2 + m_0^2c^4$$

The photon has no rest mass.